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3-Lie bialgebras of type (L_b, C_c)

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Abstract

Four dimensional 3-Lie coalgebras with two-dimensional derived algebras, and four-dimensional 3-Lie bialgebras of type (L_b, C_c) are classified. It is proved that there exist three classes of four dimensional 3-Lie coalgebras with two-dimensional derived algebra which are (L, C_{c_i}), $i = 1, 2, 3$ (Lemma 3.1), and ten classes of four dimensional 3-Lie bialgebras of type (L_b, C_c) (Theorem 3.2).

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1 Introduction

Lie bialgebra (cf [3]) has wide applications in mathematics and mathematical physics. It has close relation with solutions of classical Yang-Baxter equation. Authors in paper [1] introduced 3-Lie coalgebras and 3-Lie bialgebras, and discussed basic structures. In paper [2], 4-dimensional 3-Lie bialgebras of type (L_b, C_b) are classified. It is proved that there exist seven classes of 4-dimensional 3-Lie bialgebras of type (L_b, C_b). In this paper, we classify 3-Lie bialgebras of type (L_b, C_c), that is, the 3-Lie algebras with one-dimensional derived algebra and 3-Lie coalgebras with two-dimensional derived algebra, which we denote it by (L_b, C_c). And we suppose that 3-Lie algebras and 3-Lie coalgebras are over a field F of characteristic zero, and omit the zero multiplication of basis vectors in 3-Lie algebras and 3-Lie coalgebras.

2 Preliminaries

Let (L,μ) be a 3-**Lie algebra** [4], where $\mu: L\wedge L\wedge L\to L$ be the multiplication. Define linear mapps

$\omega_i: L\otimes L\otimes L\otimes L\otimes L\to L\otimes L\otimes L\otimes L\otimes L,\ 1\le i\le 3$, by

$\omega_1(x_1\otimes x_2\otimes x_3\otimes x_4\otimes x_5)=x_3\otimes x_4\otimes x_1\otimes x_2\otimes x_5,$

$\omega_2(x_1\otimes x_2\otimes x_3\otimes x_4\otimes x_5)=x_4\otimes x_5\otimes x_1\otimes x_2\otimes x_3,$

$\omega_3(x_1\otimes x_2\otimes x_3\otimes x_4\otimes x_5)=x_5\otimes x_3\otimes x_1\otimes x_2\otimes x_4.$

A 3-Lie coalgebra (L,Δ) [1] is a vector space L with a linear map $\Delta: L\to L\otimes L\otimes L$ satisfying

$Im(\Delta)\subset L\wedge L\wedge L$, and $(1-\omega_1-\omega_2-\omega_3)(1\otimes 1\otimes\Delta)\Delta=0,$

where $1: L^{\otimes 5}\to L^{\otimes 5}$ is identity.

A 3-Lie bialgebra[1] is a triple (L,μ,Δ) such that

(1) (L,μ) is a 3-Lie algebra with the multiplication $\mu: L\wedge L\wedge L\to L$,

(2) (L,Δ) is a 3-Lie coalgebra with $\Delta: L\to L\wedge L\wedge L$,

(3) Δ and μ satisfy the following identity, for $x,y,u,v,w\in L$,

$\Delta\mu(x,y,z)=ad_\mu^{(3)}(x,y)\Delta(z)+ad_\mu^{(3)}(y,z)\Delta(x)+ad_\mu^{(3)}(z,x)\Delta(y),$

where $ad_\mu^{(3)}(x,y)$, $ad_\mu^{(3)}(z,x)$, $ad_\mu^{(3)}(y,z): L\otimes L\otimes L\to L\otimes L\otimes L$ are linear maps defined by (similar for $ad_\mu^{(3)}(z,x)$ and $ad_\mu^{(3)}(y,z)$)

$ad_\mu^{(3)}(x,y)(u\otimes v\otimes w)=(ad_\mu(x,y)\otimes 1\otimes 1)(u\otimes v\otimes w)$

$+(1\otimes ad_\mu(x,y)\otimes 1)(u\otimes v\otimes w)+(1\otimes 1\otimes ad_\mu(x,y))(u\otimes v\otimes w)$

$=\mu(x,y,u)\otimes v\otimes w+u\otimes\mu(x,y,v)\otimes w+u\otimes v\otimes\mu(x,y,w).$

Two 3-Lie bialgebras (L_1,μ_1,Δ_1) and (L_2,μ_2,Δ_2) are called **equivalent** if there exists a linear isomorphism $f: L_1\to L_2$ such that

(1) $f:(L_1,\mu_1)\to(L_2,\mu_2)$ is a 3-Lie algebra isomorphism,

(2) $f:(L_1,\Delta_1)\to(L_2,\Delta_2)$ is a 3-Lie coalgebra isomorphism, that is,

$\Delta_2(f(x))=(f\otimes f\otimes f)\Delta_1(x)$ for all $x\in L_1$.

Lemma 2.1[4] *Let (L,μ) be a 4-dimensional 3-Lie algebra with $\dim L^1\le 2$, and e_1,e_2,e_3,e_4 be a basis of L. Then L is isomorphic to one and only one of the following: L_{b_1}. $\mu(e_2,e_3,e_4)=e_1$; L_{b_2}. $\mu(e_1,e_2,e_3)=e_1$;*

L_{c_1}. $\mu(e_2,e_3,e_4)=e_1,\mu(e_1,e_3,e_4)=e_2$;

L_{c_2}. $\mu(e_2,e_3,e_4)=\alpha e_1+e_2,\ \mu(e_1,e_3,e_4)=e_2,\ \alpha\in F,\alpha\ne 0$;

L_{c_3}. $\mu(e_1,e_3,e_4)=e_1,\ \mu(e_2,e_3,e_4)=e_2$.

3 3-Lie bialgebras of type (L_b, C_c)

We first give the classification of 3-Lie coalgebras of the type (L,C_c).

Lemma 3.1 *Let (L,Δ) be a 4-dimensional 3-Lie coalgebra with a basis e^1, e^2,e^3,e^4, and the dimension of derived algebra is 2. Then L isomorphic to one and only one of the following*

C_{c_1}. $\Delta_{c_1}(e^1)=e^2\wedge e^3\wedge e^4,\quad \Delta_{c_1}(e^2)=e^1\wedge e^3\wedge e^4$;

C_{c_2}. $\Delta_{c_2}(e^1)=\alpha e^2\wedge e^3\wedge e^4,\ \Delta_{c_2}(e^2)=e^2\wedge e^3\wedge e^4+e^1\wedge e^3\wedge e^4$;

C_{c_3}. $\Delta_{c_3}(e^1) = e^1 \wedge e^3 \wedge e^4, \quad \Delta_{c_3}(e^2) = e^2 \wedge e^3 \wedge e^4$; $\alpha \in F, \alpha \neq 0$.

Proof The result follows from Lemma 2.1 and a direct computation, we omit the computation process.

For convenience, in the following, for a 3-Lie bialgebra (L, μ, Δ), if the 3-Lie algebra (L, μ) is the case (L, μ_{b_i}) in Lemma 2.1 and the 3-Lie coalgebra (L, Δ) is the case (L, Δ_{c_j}) in Lemma 3.1, then the 3-Lie bialgebra $(L, \mu_{b_i}, \Delta_{c_j})$ is simply denoted by (L_{b_i}, C_{b_j}). *The 3-Lie bialgebras (L_{b_i}, C_{c_j}), for $1 = 1, 2; 1 \leq j \leq 3$ are called the 3-Lie bialgebras of type (L_b, C_c).*

For a given 3-Lie algebra L, in order to find all the 3-Lie bialgebra structures on L, we should find all the 3-Lie coalgebra structures on L which are compatible with the 3-Lie algebra L. Although a permutation of a basis of L gives isomorphic 3-Lie coalgebra, but it may lead to the non-equivalent 3-Lie bialgebra.

Theorem 3.2 *The non-equivalent 3-Lie bialgebras of types (L_{b_i}, C_{c_j}) for $i = 1, 2$; $j = 1, 2, 3$ are only as follows, for $\alpha \in F, \alpha \neq 0$,*

$(L_{b_1}, C_{c_1}, \Delta_1)$. $\Delta_1(e_2) = e_3 \wedge e_4 \wedge e_1, \Delta_1(e_3) = e_2 \wedge e_4 \wedge e_1$;
$(L_{b_2}, C_{c_1}, \Delta_1)$. $\Delta_1(e_2) = e_3 \wedge e_4 \wedge e_1, \Delta_1(e_3) = e_2 \wedge e_4 \wedge e_1$;
$(L_{b_1}, C_{c_3}, \Delta_5)$. $\Delta_5(e_1) = e_1 \wedge e_3 \wedge e_4, \Delta_5(e_2) = e_2 \wedge e_3 \wedge e_4$;
$(L_{b_1}, C_{c_3}, \Delta_6)$. $\Delta_6(e_1) = e_1 \wedge e_4 \wedge e_3, \Delta_6(e_2) = e_2 \wedge e_4 \wedge e_3$;
$(L_{b_1}, C_{c_3}, \Delta_7)$. $\Delta_7(e_2) = e_2 \wedge e_4 \wedge e_1, \Delta_7(e_3) = e_3 \wedge e_4 \wedge e_1$;
$(L_{b_2}, C_{c_3}, \Delta_7)$. $\Delta_7(e_2) = e_2 \wedge e_4 \wedge e_1, \Delta_7(e_3) = e_3 \wedge e_4 \wedge e_1$;
$(L_{b_1}, C_{c_2}, \Delta_2)$. $\Delta_2(e_2) = \alpha e_3 \wedge e_1 \wedge e_4, \Delta_2(e_3) = e_3 \wedge e_1 \wedge e_4 + e_2 \wedge e_1 \wedge e_4$;
$(L_{b_1}, C_{c_2}, \Delta_3)$. $\Delta_3(e_2) = \alpha e_3 \wedge e_4 \wedge e_1, \Delta_3(e_3) = e_3 \wedge e_4 \wedge e_1 + e_2 \wedge e_4 \wedge e_1$;
$(L_{b_2}, C_{c_2}, \Delta_2)$. $\Delta_2(e_2) = \alpha e_3 \wedge e_1 \wedge e_4, \Delta_2(e_3) = e_3 \wedge e_1 \wedge e_4 + e_2 \wedge e_1 \wedge e_4$;
$(L_{b_2}, C_{c_2}, \Delta_4)$. $\Delta_4(e_2) = e_2 \wedge e_4 \wedge e_1 + e_3 \wedge e_4 \wedge e_1, \Delta_4(e_3) = \alpha e_2 \wedge e_4 \wedge e_1$.

Proof First, by Lemma 2.1 and Lemma 3.1 we have to verify that whether the following twelve isomorphic 3-Lie coalgebras (obtained by permuting the basis e_1, e_2, e_3, e_4) of the type C_{c_1} are compatible with the 3-Lie algebras L_{b_1} and L_{b_2}, respectively,

(1) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_1 \wedge e_3 \wedge e_4,\ \Delta(e_3) = \Delta(e_4) = 0$;
(2) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_1 \wedge e_4 \wedge e_3,\ \Delta(e_3) = \Delta(e_4) = 0$;
(3) $\Delta(e_1) = e_3 \wedge e_2 \wedge e_4,\ \Delta(e_3) = e_1 \wedge e_2 \wedge e_4,\ \Delta(e_2) = \Delta(e_4) = 0$;
(4) $\Delta(e_1) = e_3 \wedge e_4 \wedge e_2,\ \Delta(e_3) = e_1 \wedge e_4 \wedge e_2,\ \Delta(e_2) = \Delta(e_4) = 0$;
(5) $\Delta(e_1) = e_4 \wedge e_3 \wedge e_2,\ \Delta(e_4) = e_1 \wedge e_3 \wedge e_2,\ \Delta(e_2) = \Delta(e_3) = 0$;
(6) $\Delta(e_1) = e_4 \wedge e_2 \wedge e_3,\ \Delta(e_4) = e_1 \wedge e_2 \wedge e_3,\ \Delta(e_2) = \Delta(e_3) = 0$;
(7) $\Delta(e_2) = e_3 \wedge e_4 \wedge e_1,\ \Delta(e_3) = e_2 \wedge e_4 \wedge e_1,\ \Delta(e_1) = \Delta(e_4) = 0$;
(8) $\Delta(e_2) = e_3 \wedge e_1 \wedge e_4,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4,\ \Delta(e_1) = \Delta(e_4) = 0$;
(9) $\Delta(e_2) = e_4 \wedge e_3 \wedge e_1,\ \Delta(e_4) = e_2 \wedge e_3 \wedge e_1,\ \Delta(e_1) = \Delta(e_3) = 0$;
(10) $\Delta(e_2) = e_4 \wedge e_1 \wedge e_3,\ \Delta(e_4) = e_2 \wedge e_1 \wedge e_3,\ \Delta(e_1) = \Delta(e_3) = 0$;
(11) $\Delta(e_3) = e_4 \wedge e_1 \wedge e_2,\ \Delta(e_4) = e_3 \wedge e_1 \wedge e_2,\ \Delta(e_1) = \Delta(e_2) = 0$;
(12) $\Delta(e_3) = e_4 \wedge e_2 \wedge e_1,\ \Delta(e_4) = e_3 \wedge e_2 \wedge e_1,\ \Delta(e_1) = \Delta(e_2) = 0$.

By a direct computation, only the cases (7), (8), (9), (10), (11) and (12) are compatible with the 3-Lie algebra L_{b_1}, and cases (7) and (8) are compatible with the 3-Lie algebra L_{b_2}. Define linear maps $f : L \to L$, by

(L_{b_1}, C_{c_1}). (7) $\to$ (8) : $f(e_1) = -e_1, f(e_2) = e_3,\ f(e_3) = e_2, f(e_4) = e_4$;
(9) $\to$ (10) : $f(e_1) = -e_1, f(e_2) = e_4,\ f(e_3) = e_3, f(e_4) = e_2$;

(11) $\to$ (12) and (7) $\to$ (10) : $f(e_1) = -e_1, f(e_2) = e_2, f(e_3) = e_4,\ f(e_4) = e_3$;

(8) $\to$ (11) : $f(e_1) = e_1, f(e_2) = e_3, f(e_3) = e_4, f(e_4) = e_2$.

(L_{b_2}, C_{c_1}). (7) $\to$ (8) : $f(e_1) = e_1, f(e_2) = e_3, f(e_3) = -e_2, f(e_4) = e_4$,
the non equivalent 3-Lie bialgebras of types (L_{b_j}, C_{c_1}), $j = 1, 2$, are only $(L_{b_1}, C_{c_1}, \Delta_1)$ and $(L_{b_2}, C_{c_1}, \Delta_1)$.

Second, we verify that whether the following isomorphic 3-Lie coalgebras of the type C_{c_2} are compatible with the 3-Lie algebras L_{b_1} and L_{b_2}, respectively:

(1) $\Delta(e_1) = \alpha e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_2 \wedge e_3 \wedge e_4 + e_1 \wedge e_3 \wedge e_4,\ \Delta(e_3) = \Delta(e_4) = 0$;
(2) $\Delta(e_1) = \alpha e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_2 \wedge e_4 \wedge e_3 + e_1 \wedge e_4 \wedge e_3,\ \Delta(e_3) = \Delta(e_4) = 0$;
(3) $\Delta(e_1) = e_1 \wedge e_3 \wedge e_4 + e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = \alpha e_1 \wedge e_3 \wedge e_4,\ \Delta(e_3) = \Delta(e_4) = 0$;
(4) $\Delta(e_1) = e_1 \wedge e_4 \wedge e_3 + e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = \alpha e_1 \wedge e_4 \wedge e_3,\ \Delta(e_3) = \Delta(e_4) = 0$;
(5) $\Delta(e_1) = e_1 \wedge e_2 \wedge e_4 + e_3 \wedge e_2 \wedge e_4,\ \Delta(e_3) = \alpha e_1 \wedge e_2 \wedge e_4,\ \Delta(e_2) = \Delta(e_4) = 0$;
(6) $\Delta(e_1) = e_1 \wedge e_4 \wedge e_2 + e_3 \wedge e_4 \wedge e_2,\ \Delta(e_3) = \alpha e_1 \wedge e_4 \wedge e_2,\ \Delta(e_2) = \Delta(e_4) = 0$;
(7) $\Delta(e_1) = \alpha e_3 \wedge e_4 \wedge e_2,\ \Delta(e_3) = e_3 \wedge e_4 \wedge e_2 + e_1 \wedge e_4 \wedge e_2,\ \Delta(e_2) = \Delta(e_4) = 0$;
(8) $\Delta(e_1) = \alpha e_3 \wedge e_2 \wedge e_4,\ \Delta(e_3) = e_3 \wedge e_2 \wedge e_4 + e_1 \wedge e_2 \wedge e_4,\ \Delta(e_2) = \Delta(e_4) = 0$;
(9) $\Delta(e_1) = e_1 \wedge e_3 \wedge e_2 + e_4 \wedge e_3 \wedge e_2,\ \Delta(e_4) = \alpha e_1 \wedge e_3 \wedge e_2,\ \Delta(e_2) = \Delta(e_3) = 0$;
(10) $\Delta(e_1) = e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_2 \wedge e_3,\ \Delta(e_4) = \alpha e_1 \wedge e_2 \wedge e_3,\ \Delta(e_2) = \Delta(e_3) = 0$;
(11) $\Delta(e_1) = \alpha e_4 \wedge e_2 \wedge e_3,\ \Delta(e_4) = e_4 \wedge e_2 \wedge e_3 + e_1 \wedge e_2 \wedge e_3,\ \Delta(e_2) = \Delta(e_3) = 0$;
(12) $\Delta(e_1) = \alpha e_4 \wedge e_3 \wedge e_2,\ \Delta(e_4) = e_4 \wedge e_3 \wedge e_2 + e_1 \wedge e_3 \wedge e_2,\ \Delta(e_2) = \Delta(e_3) = 0$;
(13) $\Delta(e_2) = \alpha e_3 \wedge e_1 \wedge e_4,\ \Delta(e_3) = e_3 \wedge e_1 \wedge e_4 + e_2 \wedge e_1 \wedge e_4,\ \Delta(e_1) = \Delta(e_4) = 0$;
(14) $\Delta(e_2) = \alpha e_3 \wedge e_4 \wedge e_1,\ \Delta(e_3) = e_3 \wedge e_4 \wedge e_1 + e_2 \wedge e_4 \wedge e_1,\ \Delta(e_1) = \Delta(e_4) = 0$;
(15) $\Delta(e_2) = e_2 \wedge e_4 \wedge e_1 + e_3 \wedge e_4 \wedge e_1,\ \Delta(e_3) = \alpha e_2 \wedge e_4 \wedge e_1,\ \Delta(e_1) = \Delta(e_4) = 0$;
(16) $\Delta(e_2) = e_2 \wedge e_1 \wedge e_4 + e_3 \wedge e_1 \wedge e_4,\ \Delta(e_3) = \alpha e_2 \wedge e_1 \wedge e_4,\ \Delta(e_1) = \Delta(e_4) = 0$;
(17) $\Delta(e_2) = \alpha e_4 \wedge e_3 \wedge e_1,\ \Delta(e_4) = e_4 \wedge e_3 \wedge e_1 + e_2 \wedge e_3 \wedge e_1,\ \Delta(e_1) = \Delta(e_3) = 0$;
(18) $\Delta(e_2) = \alpha e_4 \wedge e_1 \wedge e_3,\ \Delta(e_4) = e_4 \wedge e_1 \wedge e_3 + e_2 \wedge e_1 \wedge e_3,\ \Delta(e_1) = \Delta(e_3) = 0$;
(19) $\Delta(e_2) = e_2 \wedge e_1 \wedge e_3 + e_4 \wedge e_1 \wedge e_3,\ \Delta(e_4) = \alpha e_2 \wedge e_1 \wedge e_3,\ \Delta(e_1) = \Delta(e_3) = 0$;
(20) $\Delta(e_2) = e_2 \wedge e_3 \wedge e_1 + e_4 \wedge e_3 \wedge e_1,\ \Delta(e_4) = \alpha e_2 \wedge e_3 \wedge e_1,\ \Delta(e_1) = \Delta(e_3) = 0$;
(21) $\Delta(e_3) = \alpha e_4 \wedge e_1 \wedge e_2,\ \Delta(e_4) = e_4 \wedge e_1 \wedge e_2 + e_3 \wedge e_1 \wedge e_2,\ \Delta(e_1) = \Delta(e_2) = 0$;
(22) $\Delta(e_3) = \alpha e_4 \wedge e_2 \wedge e_1,\ \Delta(e_4) = e_4 \wedge e_2 \wedge e_1 + e_3 \wedge e_2 \wedge e_1,\ \Delta(e_1) = \Delta(e_2) = 0$;
(23) $\Delta(e_3) = e_3 \wedge e_2 \wedge e_1 + e_4 \wedge e_2 \wedge e_1,\ \Delta(e_4) = \alpha e_3 \wedge e_2 \wedge e_1,\ \Delta(e_1) = \Delta(e_2) = 0$;
(24) $\Delta(e_3) = e_3 \wedge e_1 \wedge e_2 + e_4 \wedge e_1 \wedge e_2,\ \Delta(e_4) = \alpha e_3 \wedge e_1 \wedge e_2,\ \Delta(e_1) = \Delta(e_2) = 0$.

By a direct computation, only the cases (13), (14), (15), (16), (17), (18), (19), (20), (21), (22), (23) and (24) are compatible with the 3-Lie algebra L_{b_1}, and cases (13), (14), (15) and (16) are compatible with the 3-Lie algebra L_{b_2}.

By the algebra isomorphisms $f : L \to L$ of (L_{b_1}, C_{c_2}): (13) $\to$ (15) and (14) $\to$ (16) : $f(e_1) = -e_1, f(e_2) = e_3, f(e_3) = e_2, f(e_4) = e_4$; (17) $\to$ (19) and (18) $\to$ (20) : $f(e_1) = -e_1, f(e_2) = e_4, f(e_3) = e_3, f(e_4) = e_2$; (13) $\to$ (17), (14) $\to$ (18), (21) $\to$ (23) and (22) $\to$ (24) : $f(e_1) = -e_1, f(e_2) = e_2, f(e_3) = e_4, f(e_4) = e_3$; (13) $\to$ (21) and (14) $\to$ (22) : $f(e_1) = e_1, f(e_2) = e_3, f(e_3) = e_4, f(e_4) = e_2$; we obtain 3-Lie bialgebras of cases (13), (15), (17), (19), (21) and (23) are equivalent, and the cases (14), (16), (18), (20), (22) and (24) are equivalent.

If $h : L_{b_1} \to L_{b_1}$ is a 3-Lie algebra isomorphism, then we have $h(e_1) = \lambda e_1$ for $\lambda \in F$ and $\lambda \neq 0$. If h satisfies $h \otimes h \otimes h(\triangle_2(e_2)) = \Delta_3(h(e_2))$, and $h \otimes h \otimes h(\Delta_2(e_3)) = \Delta_3(h(e_2))$, then by a direct computation, h is degenerate. Therefore, 3-Lie bialgebras $(L_{b_1}, C_{c_2}, \Delta_2)$, $(L_{b_1}, C_{c_2}, \Delta_3)$ are non-equivalent.

From the algebra isomorphisms $f : L \to L$ of (L_{b_2}, C_{c_2}): (13) $\to$ (14) and (15) $\to$ (16) : $f(e_1) = e_1, f(e_2) = e_2, f(e_3) = e_3, f(e_4) = -e_4$; and the similar discussion, $(L_{b_2}, C_{c_2}, \Delta_4)$ are non-equivalent with $(L_{b_2}, C_{c_2}, \Delta_2)$, and $(L_{b_2}, C_{c_2}, \Delta_3)$.

Lastly, we study wether the following 3-Lie coalgebras of the type C_{c_3} are compatible with the Lie algebra L_{b_1} and L_{b_2}, respectively,

(1) $\Delta(e_1) = e_1 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_2 \wedge e_3 \wedge e_4,\ \Delta(e_3) = \Delta(e_4) = 0;$
(2) $\Delta(e_1) = e_1 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_3) = \Delta(e_4) = 0;$
(3) $\Delta(e_1) = e_1 \wedge e_2 \wedge e_4,\ \Delta(e_3) = e_3 \wedge e_2 \wedge e_4,\ \Delta(e_2) = \Delta(e_4) = 0;$
(4) $\Delta(e_1) = e_1 \wedge e_4 \wedge e_2,\ \Delta(e_3) = e_3 \wedge e_4 \wedge e_2,\ \Delta(e_2) = \Delta(e_4) = 0;$
(5) $\Delta(e_1) = e_1 \wedge e_3 \wedge e_2,\ \Delta(e_4) = e_4 \wedge e_3 \wedge e_2,\ \Delta(e_2) = \Delta(e_3) = 0;$
(6) $\Delta(e_1) = e_1 \wedge e_2 \wedge e_3,\ \Delta(e_4) = e_4 \wedge e_2 \wedge e_3,\ \Delta(e_2) = \Delta(e_3) = 0;$
(7) $\Delta(e_2) = e_2 \wedge e_4 \wedge e_1,\ \Delta(e_3) = e_3 \wedge e_4 \wedge e_1,\ \Delta(e_1) = \Delta e_4 = 0;$
(8) $\Delta(e_2) = e_2 \wedge e_1 \wedge e_4,\ \Delta(e_3) = e_3 \wedge e_1 \wedge e_4,\ \Delta(e_1) = \Delta e_4 = 0;$
(9) $\Delta(e_2) = e_2 \wedge e_3 \wedge e_1,\ \Delta(e_4) = e_4 \wedge e_3 \wedge e_1,\ \Delta(e_1) = \Delta e_3 = 0;$
(10) $\Delta(e_2) = e_2 \wedge e_1 \wedge e_3,\ \Delta(e_4) = e_4 \wedge e_1 \wedge e_3,\ \Delta(e_1) = \Delta(e_3) = 0;$
(11) $\Delta(e_3) = e_3 \wedge e_1 \wedge e_2,\ \Delta(e_4) = e_4 \wedge e_1 \wedge e_2,\ \Delta(e_1) = \Delta(e_2) = 0;$
(12) $\Delta(e_3) = e_3 \wedge e_2 \wedge e_1,\ \Delta(e_4) = e_4 \wedge e_2 \wedge e_1,\ \Delta(e_1) = \Delta(e_2) = 0.$

By a direct computation, all twelve cases are compatible with the 3-Lie algebra L_{b_1}. And the only cases (7) and (8) are compatible with the 3-Lie algebra L_{b_2}.

By the similar discussions to the cases $(L_{\mu_{b_i}}, C_{c_2})$ for $i = 1, 2$ and isomorphisms of 3-Lie bialgebras of (L_{b_1}, C_{c_3}): (1) $\to$ (3), (2) $\to$ (4) and (7) $\to$ (8) : $f(e_1) = -e_1, f(e_2) = e_3, f(e_3) = e_2, f(e_4) = e_4$; (1) $\to$ (5), (2) $\to$ (6) and (7) $\to$ (11) : $f(e_1) = -e_1, f(e_2) = e_4, f(e_3) = e_3, f(e_4) = e_2$; (7) $\to$ (9) and (8) $\to$ (10) : $f(e_1) = e_1, f(e_2) = -e_2, f(e_3) = e_4, f(e_4) = e_3$; (11) $\to$ (12) : $f(e_1) = -e_1, f(e_2) = e_2, f(e_3) = e_4, f(e_4) = e_3$. (L_{b_2}, C_{c_3}). (7) $\to$ (8) : $f(e_1) = -e_1, f(e_2) = e_2, f(e_3) = e_3, f(e_4) = e_4$, and the similar discussion to the case $(L_{\mu_{b_1}}, C_{c_2})$ and $(L_{\mu_{b_2 1}}, C_{c_2})$, we get non-equivalent 3-Lie bialgebras $(L_{b_1}, C_{c_3}, \Delta_i)$ for $i = 5, 6, 7$ and $(L_{b_2}, C_{c_3}, \Delta_7)$. The proof is complete.

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