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3-Lie bialgebras (L_b, C_d) and (L_b, C_e)

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Abstract

In this paper, we continue to study the structure of four dimensional 3-Lie bialgebras. We discuss the existence of 3-Lie bialgebras of types (L_b, C_d) and (L_b, C_e). It is proved that there do not exist 3-Lie bialgebras of types (L_{b_1}, C_e), (L_{b_2}, C_e) and (L_{b_2}, C_d). There exists only one class of 3-Lie bialgebras of type (L_b, C_d), that is (L_{b_1}, C_d, Δ_1) (Theorem 3.4).

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1 Introduction

In 1985, Filippov provided n-Lie algebras [1]. Since then, the n-Lie algebra, especially, the 3-Lie algebra attracts more and more attention, and it is widely used in mathematics, mathematical physics and string theory. And some n-ary algebras such as n-Hom algebras, n-supper algebras, n-Rota-Baxter algebras, etc. are provided and studied (see [2, 3, 4]). Authors in paper [5] introduced 3-Lie coalgebras and 3-Lie bialgebras, and constructed some 4-dimensional 3-Lie bialgebras. In paper [6], 4-dimensional 3-Lie bialgebras of type (L_b, C_b) are classified. It is proved that there exist seven classes of 4-dimensional 3-Lie bialgebras of type (L_b, C_b). In this paper, we discuss 3-Lie bialgebras of types (L_b, C_d) and (L_b, C_e), that is, the 3-Lie algebras with one-dimensional derived algebra and 3-Lie coalgebras with three and four dimensional derived algebra, which is denoted by (L_b, C_d) and (L_b, C_e), respectively. And we suppose that 3-Lie algebras and 3-Lie coalgebras are over a field F of characteristic zero, and omit the zero multiplication of basis vectors in 3-Lie algebras and 3-Lie coalgebras.

2 Preliminaries

A 3-Lie algebra [1] is a vector space L endowed with a linear multiplication $\mu : L^{\wedge 3} \to L$ satisfying that, for all $x, y, z, u, v \in L$,

$\mu(u, v, \mu(x, y, z)) = \mu(x, y, \mu(u, v, z)) + \mu(y, z, \mu(u, v, x)) + \mu(z, x, \mu(u, v, y))$.

For defining 3-Lie coalgebras, we need to define following linear mapps

$\omega_i : L \otimes L \otimes L \otimes L \otimes L \to L \otimes L \otimes L \otimes L \otimes L,\ 1 \leq i \leq 3$, by

$\omega_1(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_3 \otimes x_4 \otimes x_1 \otimes x_2 \otimes x_5,$

$\omega_2(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_4 \otimes x_5 \otimes x_1 \otimes x_2 \otimes x_3,$

$\omega_3(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_5 \otimes x_3 \otimes x_1 \otimes x_2 \otimes x_4.$

A 3-Lie coalgebra (L, Δ) [5] is a vector space L with a linear map $\Delta : L \to L \otimes L \otimes L$ satisfying

$Im(\Delta) \subset L \wedge L \wedge L$, and $(1 - \omega_1 - \omega_2 - \omega_3)(1 \otimes 1 \otimes \Delta)\Delta = 0,$

where $1 : L^{\otimes 5} \to L^{\otimes 5}$ is identity.

Let (L_1, Δ_1) and (L_2, Δ_2) be 3-Lie coalgebras. If there is a linear isomorphism $\varphi : L_1 \to L_2$ satisfying $(\varphi \otimes \varphi \otimes \varphi)(\Delta_1(e)) = \Delta_2(\varphi(e))$, for all $e \in L_1$, then (L_1, Δ_1) is isomorphic to (L_2, Δ_2), and φ is called *a 3-Lie coalgebra isomorphism*, where $(\varphi \otimes \varphi \otimes \varphi) \sum\limits_i (a_i \otimes b_i \otimes c_i) = \sum\limits_i \varphi(a_i) \otimes \varphi(b_i) \otimes \varphi(c_i)$.

A 3-Lie bialgebra [5] is a triple (L, μ, Δ) such that

(1) (L, μ) is a 3-Lie algebra with the multiplication $\mu : L \wedge L \wedge L \to L$,

(2) (L, Δ) is a 3-Lie coalgebra with $\Delta : L \to L \wedge L \wedge L$,

(3) Δ and μ satisfy the following identities, for $x, y, u, v, w \in L$,

$\Delta\mu(x, y, z) = ad_\mu^{(3)}(x, y)\Delta(z) + ad_\mu^{(3)}(y, z)\Delta(x) + ad_\mu^{(3)}(z, x)\Delta(y),$

where $ad_\mu^{(3)}(x, y)$, $ad_\mu^{(3)}(z, x)$, $ad_\mu^{(3)}(y, z) : L \otimes L \otimes L \to L \otimes L \otimes L$ are linear maps defined by (similar for $ad_\mu^{(3)}(z, x)$ and $ad_\mu^{(3)}(y, z)$)

$ad_\mu^{(3)}(x, y)(u \otimes v \otimes w) = (ad_\mu(x, y) \otimes 1 \otimes 1)(u \otimes v \otimes w)$

$+(1 \otimes ad_\mu(x, y) \otimes 1)(u \otimes v \otimes w) + (1 \otimes 1 \otimes ad_\mu(x, y))(u \otimes v \otimes w)$

$= \mu(x, y, u) \otimes v \otimes w + u \otimes \mu(x, y, v) \otimes w + u \otimes v \otimes \mu(x, y, w).$

Two 3-Lie bialgebras (L_1, μ_1, Δ_1) and (L_2, μ_2, Δ_2) are called **equivalent** if there exists a linear isomorphism $f : L_1 \to L_2$ such that

(1) $f : (L_1, \mu_1) \to (L_2, \mu_2)$ is a 3-Lie algebra isomorphism,

(2) $f : (L_1, \Delta_1) \to (L_2, \Delta_2)$ is a 3-Lie coalgebra isomorphism, that is,

$\Delta_2(f(x)) = (f \otimes f \otimes f)\Delta_1(x)$ for all $x \in L_1$.

Lemma 2.1[1] *Let (L, μ) be a 4-dimensional 3-Lie algebra with $\dim L^1 \neq 0, 2$, and e_1, e_2, e_3, e_4 be a basis of L. Then L is isomorphic to one and only one of the following*

L_{b_1}. $\mu(e_2, e_3, e_4) = e_1$. L_{b_2}. $\mu(e_1, e_2, e_3) = e_1$.
L_d. $\mu_d(e_2, e_3, e_4) = e_1, \mu_d(e_1, e_3, e_4) = e_2, \mu_d(e_1, e_2, e_4) = e_3$.
L_e. $\mu_e(e_2, e_3, e_4) = e_1, \mu_e(e_1, e_3, e_4) = e_2, \mu_e(e_1, e_2, e_4) = e_3, \mu_e(e_1, e_2, e_3) = e_4$.

Lemma 2.2 [5] *Let L be a vector space over F, and $\mu : L \otimes L \otimes L \to L$ be a 3-ary linear map. Then (L, μ) is a 3-Lie algebra if and only if (L^*, μ^*) is a 3-Lie coalgebra with $\mu^* : L^* \to L^* \otimes L^* \otimes L^*$, where μ^* is the dual map of μ.*

3 3-Lie bialgebras of types (L_b, C_d) and (L_b, C_e)

First We give the classification of 3-Lie coalgebras of the types (L, C_d) and (L, C_e).

Lemma 3.1 *Let (L, Δ) be a 4-dimensional 3-Lie coalgebra with m-dimensional derived algebra ($3 \le m \le 4$), and e^1, e^2, e^3, e^4 be a basis of L. Then L isomorphic to one and only one of the following*
C_d. $\Delta_d(e^1) = e^2 \wedge e^3 \wedge e^4, \Delta_d(e^2) = e^1 \wedge e^3 \wedge e^4, \Delta_d(e^3) = e^1 \wedge e^2 \wedge e^4$;
C_e. $\Delta_e(e^1) = e^2 \wedge e^3 \wedge e^4, \Delta_e(e^2) = e^1 \wedge e^3 \wedge e^4, \Delta_e(e^3) = e^1 \wedge e^2 \wedge e^4$,
$\Delta_e(e^4) = e^1 \wedge e^2 \wedge e^3$.

Proof The result follows from Lemma 2.1, Lemma 2.2 and a direct computation, we omit the computation process.

For convenience, in the following, for a 3-Lie bialgebra (L, μ, Δ), if the 3-Lie algebra (L, μ) is the case (L, μ_{b_i}) in Lemma 2.1 and the 3-Lie coalgebra (L, Δ) is the case (L, Δ_d) and (L, Δ_e) in Lemma 3.1, then the 3-Lie bialgebra (L, μ_{b_i}, Δ_d) and (L, μ_{b_i}, Δ_e) are simply denoted by (L_{b_i}, C_d) and (L_{b_i}, C_e), which are called *the 3-Lie bialgebras of type (L_b, C_d), and* (L_b, C_e), respectively.

For a given 3-Lie algebra L, in order to find all the 3-Lie bialgebra structures on L, we should find all the 3-Lie coalgebra structures on L which are compatible with the 3-Lie algebra L. Although a permutation of a basis of L gives isomorphic 3-Lie coalgebra, but it may lead to the non-equivalent 3-Lie bialgebra.

Theorem 3.2 *There do not exist 3-Lie bialgebras of the types (L_{b_1}, C_e) and (L_{b_2}, C_e).*

Proof By Lemma 2.1 and 2.2, we need to verify that 3-Lie algebras L_{b_1} and L_{b_2} are incompatible with the following six isomorphic 3-Lie coalgebras of the type C_e:
(1).$\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_1 \wedge e_3 \wedge e_4, \Delta(e_3) = e_1 \wedge e_2 \wedge e_4, \Delta(e_4) = e_1 \wedge e_2 \wedge e_3$;
(2).$\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_1 \wedge e_3 \wedge e_4, \Delta(e_3) = e_2 \wedge e_1 \wedge e_4, \Delta(e_4) = e_2 \wedge e_1 \wedge e_3$;
(3).$\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_3 \wedge e_1 \wedge e_4, \Delta(e_3) = e_2 \wedge e_1 \wedge e_4, \Delta(e_4) = e_2 \wedge e_3 \wedge e_1$;
(4).$\Delta(e_1) = e_2 \wedge e_4 \wedge e_3, \Delta(e_2) = e_1 \wedge e_4 \wedge e_3, \Delta(e_3) = e_2 \wedge e_1 \wedge e_4, \Delta(e_4) = e_2 \wedge e_1 \wedge e_3$;
(5).$\Delta(e_1) = e_2 \wedge e_4 \wedge e_3, \Delta(e_2) = e_4 \wedge e_1 \wedge e_3, \Delta(e_3) = e_2 \wedge e_4 \wedge e_1, \Delta(e_4) = e_2 \wedge e_1 \wedge e_3$;
(6).$\Delta(e_1) = e_2 \wedge e_4 \wedge e_3, \Delta(e_2) = e_4 \wedge e_3 \wedge e_1, \Delta(e_3) = e_2 \wedge e_4 \wedge e_1, \Delta(e_4) = e_2 \wedge e_3 \wedge e_1$.

Here we only check the case (1) is incompatible with the 3-Lie algebra L_{b_1}. Since
$\Delta\mu_{b_1}(e_2, e_3, e_4) = \Delta(e_1) = e_2 \wedge e_3 \wedge e_4$, but
$ad^{(3)}_{\mu_{b_1}}(e_2, e_3)\Delta(e_4) + ad^{(3)}_{\mu_{b_1}}(e_3, e_4)\Delta(e_2) + ad^{(3)}_{\mu_{b_1}}(e_4, e_2)\Delta(e_3) = 0$,
we obtain that
$\Delta\mu_{b_1}e_2, e_3, e_4) \neq ad^{(3)}_{\mu_{b_1}}(e_2, e_3)\Delta(e_4) + ad^{(3)}_{\mu_{b_1}}(e_3, e_4)\Delta(e_2) + ad^{(3)}_{\mu_{b_1}}(e_4, e_2)\Delta(e_3)$.
Therefore, the case (1) of the type C_e is incompatible with the 3-Lie algebra L_{b_1}. Similar discussions for others cases, we get the result.

Theorem 3.3 *There does not exist 3-Lie bialgebras of the type* (L_{b_2}, C_d).

Proof By Lemma 2.1 and 2.2, we need to verify that 3-Lie algebra L_{b_2} is incompatible with following twenty-four isomorphic 3-Lie coalgebras of the type C_d:

(1).$\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_1 \wedge e_3 \wedge e_4, \Delta(e_3) = e_1 \wedge e_2 \wedge e_4;$
(2).$\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_3 \wedge e_1 \wedge e_4, \Delta(e_3) = e_2 \wedge e_1 \wedge e_4;$
(3).$\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_1 \wedge e_3 \wedge e_4, \Delta(e_3) = e_2 \wedge e_1 \wedge e_4;$
(4).$\Delta(e_1) = e_3 \wedge e_2 \wedge e_4, \Delta(e_2) = e_3 \wedge e_1 \wedge e_4, \Delta(e_3) = e_1 \wedge e_2 \wedge e_4;$
(5).$\Delta(e_1) = e_3 \wedge e_2 \wedge e_4, \Delta(e_2) = e_3 \wedge e_1 \wedge e_4, \Delta(e_3) = e_2 \wedge e_1 \wedge e_4;$
(6).$\Delta(e_1) = e_3 \wedge e_2 \wedge e_4, \Delta(e_2) = e_1 \wedge e_3 \wedge e_4, \Delta(e_3) = e_1 \wedge e_2 \wedge e_4;$
(7).$\Delta(e_1) = e_2 \wedge e_4 \wedge e_3, \Delta(e_2) = e_1 \wedge e_4 \wedge e_3, \Delta(e_4) = e_2 \wedge e_1 \wedge e_3;$
(8).$\Delta(e_1) = e_2 \wedge e_4 \wedge e_3, \Delta(e_2) = e_4 \wedge e_1 \wedge e_3, \Delta(e_4) = e_2 \wedge e_1 \wedge e_3;$
(9).$\Delta(e_1) = e_2 \wedge e_4 \wedge e_3, \Delta(e_2) = e_1 \wedge e_4 \wedge e_3, \Delta(e_4) = e_1 \wedge e_2 \wedge e_3;$
(10).$\Delta(e_1) = e_4 \wedge e_2 \wedge e_3, \Delta(e_2) = e_4 \wedge e_1 \wedge e_3, \Delta(e_4) = e_2 \wedge e_1 \wedge e_3;$
(11).$\Delta(e_1) = e_4 \wedge e_2 \wedge e_3, \Delta(e_2) = e_1 \wedge e_4 \wedge e_3, \Delta(e_4) = e_1 \wedge e_2 \wedge e_3;$
(12).$\Delta(e_1) = e_4 \wedge e_2 \wedge e_3, \Delta(e_2) = e_4 \wedge e_1 \wedge e_3, \Delta(e_4) = e_1 \wedge e_2 \wedge e_3;$
(13).$\Delta(e_1) = e_3 \wedge e_4 \wedge e_2, \Delta(e_3) = e_4 \wedge e_1 \wedge e_2, \Delta(e_4) = e_3 \wedge e_1 \wedge e_2;$
(14).$\Delta(e_1) = e_3 \wedge e_4 \wedge e_2, \Delta(e_3) = e_1 \wedge e_4 \wedge e_2, \Delta(e_4) = e_1 \wedge e_3 \wedge e_2;$
(15).$\Delta(e_1) = e_3 \wedge e_4 \wedge e_2, \Delta(e_3) = e_1 \wedge e_4 \wedge e_2, \Delta(e_4) = e_3 \wedge e_1 \wedge e_2;$
(16).$\Delta(e_1) = e_4 \wedge e_3 \wedge e_2, \Delta(e_3) = e_4 \wedge e_1 \wedge e_2, \Delta(e_4) = e_1 \wedge e_3 \wedge e_2;$
(17).$\Delta(e_1) = e_4 \wedge e_3 \wedge e_2, \Delta(e_3) = e_4 \wedge e_1 \wedge e_2, \Delta(e_4) = e_3 \wedge e_1 \wedge e_2;$
(18).$\Delta(e_1) = e_4 \wedge e_3 \wedge e_2, \Delta(e_3) = e_1 \wedge e_4 \wedge e_2, \Delta(e_4) = e_1 \wedge e_3 \wedge e_2;$
(19).$\Delta(e_2) = e_4 \wedge e_3 \wedge e_1, \Delta(e_3) = e_4 \wedge e_2 \wedge e_1, \Delta(e_4) = e_3 \wedge e_2 \wedge e_1;$
(20).$\Delta(e_2) = e_4 \wedge e_3 \wedge e_1, \Delta(e_3) = e_2 \wedge e_4 \wedge e_1, \Delta(e_4) = e_3 \wedge e_2 \wedge e_1;$
(21).$\Delta(e_2) = e_4 \wedge e_3 \wedge e_1, \Delta(e_3) = e_4 \wedge e_2 \wedge e_1, \Delta(e_4) = e_2 \wedge e_3 \wedge e_1;$
(22).$\Delta(e_2) = e_3 \wedge e_4 \wedge e_1, \Delta(e_3) = e_2 \wedge e_4 \wedge e_1, \Delta(e_4) = e_3 \wedge e_2 \wedge e_1;$
(23).$\Delta(e_2) = e_3 \wedge e_4 \wedge e_1, \Delta(e_3) = e_2 \wedge e_4 \wedge e_1, \Delta(e_4) = e_2 \wedge e_3 \wedge e_1;$
(24).$\Delta(e_2) = e_3 \wedge e_4 \wedge e_1, \Delta(e_3) = e_4 \wedge e_2 \wedge e_1, \Delta(e_4) = e_3 \wedge e_2 \wedge e_1.$

The discussion is completely similar to Theorem 3.2. We omit the computing process.

Theorem 3.4 *The only non-equivalent 3-Lie bialgebras of the type* (L_{b_1}, C_d) *is*

$(L_{b_1}, C_d, \Delta_1) : \Delta_1(e_2) = e_1 \wedge e_4 \wedge e_3, \Delta_1(e_3) = e_1 \wedge e_4 \wedge e_2, \Delta_1(e_4) = e_1 \wedge e_3 \wedge e_2.$

Proof We need to discuss the compatibility of twenty-four isomorphic 3-Lie coalgebras of the type C_{c_d} in Theorem 3.3 with the 3-Lie algebra L_{b_1}.

By a direct computation, only cases (19), (20), (21), (22), (23) and (24) are compatible with the 3-Lie algebra L_{b_1}, respectively. Thanks to isomorphisms of 3-Lie bialgebras:

$(19) \to (22) : f(e_1) = -e_1, f(e_2) = e_3, f(e_3) = e_2, f(e_4) = e_4;$
$(21) \to (23), (23) \to (24) : f(e_1) = e_1, f(e_2) = e_3, f(e_3) = e_4, f(e_4) = e_2;$
$(19) \to (20) : f(e_1) = e_1, f(e_2) = -e_2, f(e_3) = \sqrt{-1}e_3, f(e_4) = \sqrt{-1}e_4;$
$(19) \to (21) : f(e_1) = e_1, f(e_2) = -e_2, f(e_3) = -\sqrt{-1}e_3, f(e_4) = -\sqrt{-1}e_4;$

we get that the only non-equivalent 3-Lie bialgebras of the type (L_{b_1}, C_d) is (L_{b_1}, C_d, Δ_1).

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