

Exponentially small expansions associated with a generalised Mathieu series

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Abstract

We consider the generalised Mathieu series

$$\sum_{n=1}^{\infty} \frac{n^{\gamma}}{(n^{\lambda} + a^{\lambda})^{\mu}} \quad (\mu > 0)$$

when the parameters $\lambda (> 0)$ and γ are both even integers for large complex a in the sector $|\arg a| < \pi/\lambda$. The asymptotics in this case consist of a *finite* algebraic expansion together with an infinite sequence of increasingly subdominant exponentially small expansions. When μ is also a positive integer it is possible to give closed-form evaluations of this series. Numerical results are given to illustrate the accuracy of the expansion obtained.

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1. Introduction

This paper is a sequel to the asymptotic study of a generalised Mathieu series carried out by the author in [4]. The functional series

$$\sum_{n=1}^{\infty} \frac{n}{(n^2 + a^2)^{\mu}} \tag{1.1}$$

in the case $\mu = 2$ was introduced by Mathieu in his 1890 book [2] dealing with the elasticity of solid bodies. Considerable effort has been devoted to the determination of upper and low bounds for the series with $\mu = 2$ when the parameter $a > 0$. Several integral representations for (1.1), and its alternating variant, have been obtained; see [8] and the references therein.

The asymptotic expansion of the more general functional series

$$S_{\mu,\gamma}(a; \lambda) := \sum_{n=1}^{\infty} \frac{n^{\gamma}}{(n^{\lambda} + a^{\lambda})^{\mu}} \quad (\mu > 0, \lambda > 0, \lambda\mu - \gamma > 1) \quad (1.2)$$

was considered by Zvastavnyi [11] for $a \rightarrow +\infty$ and by Paris [4] for $|a| \rightarrow \infty$ in the sector $|\arg a| < \pi/\lambda$. In [4], the additional factor $e_n := \exp[-a^{\lambda}b/(n^{\lambda} + a^{\lambda})]$ (with $b > 0$) was included in the summand which, although not affecting the rate of convergence of the series (since $e_n \rightarrow 1$ as $n \rightarrow \infty$), can modify the large- a growth, particularly with the alternating variant of (1.2). Both these authors adopted a Mellin transform method and obtained the result¹, when $\gamma > -1$,

$$S_{\mu,\gamma}(a; \lambda) - \frac{\Gamma(\frac{\gamma+1}{\lambda})\Gamma(\mu - \frac{\gamma+1}{\lambda})}{\lambda\Gamma(\mu)a^{\lambda\mu-\gamma-1}} \sim \frac{1}{\Gamma(\mu)} \sum_{k=0}^{\infty} \frac{(-)^k \Gamma(\mu+k)}{k! a^{\lambda(k+\mu)}} \zeta(-\gamma - \lambda k) \quad (1.3)$$

as $|a| \rightarrow \infty$ in the sector $|\arg a| < \pi/\lambda$, where $\zeta(s)$ denotes the Riemann zeta function.

In this paper we also employ the Mellin transform approach used in [4, 11], where our interest will be concerned with the parameter values $\mu > 0$ and even integer values of $\lambda (> 0)$ and γ . In this case, the asymptotic series on the right-hand side of (1.3) (the algebraic expansion) is either a finite series or vacuous on account of the trivial zeros of $\zeta(s)$. We shall find that the asymptotic expansion of $S_{\mu,\gamma}(a; \lambda)$ for large complex a in the sector $|\arg a| < \pi/\lambda$ for these parameter values consists of a finite algebraic expansion together with an infinite sequence (when μ is not an integer) of increasingly subdominant exponentially small contributions. In the case of positive integer μ it is possible to give a closed-form evaluation of $S_{\mu,\gamma}(a; \lambda)$.

It is perhaps rather surprising that such an innocent-looking series should possess such an intricate asymptotic structure. A similar phenomenon has been recently observed in the expansion of the generalised Euler-Jacobi series $\sum_{n=1}^{\infty} n^{-w} \exp[-an^p]$ as the parameter $a \rightarrow 0$ when p and w are even integers; see [5]. The leading terms in the expansion of $S_{\mu,\gamma}(a; \lambda)$ when $\gamma = 0$ and $\lambda = 2, 4$ have been given in [10] using the Poisson-Jacobi formula.

In the application of the Mellin transform method to the series in (1.2) and its alternating variant we shall require the following estimates for the gamma function and the Riemann zeta function. For real σ and t , we have the estimates

$$\Gamma(\sigma \pm it) = O(t^{\sigma-\frac{1}{2}}), \quad |\zeta(\sigma \pm it)| = O(t^{\Omega(\sigma)} \log^{\alpha} t) \quad (t \rightarrow +\infty), \quad (1.4)$$

¹The restriction $\gamma > -1$ was imposed in [4] to avoid the formation of a double pole for odd negative integer values of γ ; in [11], the parameter γ was allowed to assume arbitrary real values and $a \rightarrow +\infty$.

where $\Omega(\sigma) = 0$ ($\sigma > 1$), $\frac{1}{2} - \frac{1}{2}\sigma$ ($0 \leq \sigma \leq 1$), $\frac{1}{2} - \sigma$ ($\sigma < 0$) and $\alpha = 1$ ($0 \leq \sigma \leq 1$), $\alpha = 0$ otherwise [9, p. 95]. The zeta function $\zeta(s)$ has a simple pole of unit residue at $s = 1$ and the evaluations for positive integer k

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-2k) = 0, \quad \zeta(2k) = \frac{(2\pi)^{2k}}{2(2k)!} |B_{2k}| \quad (k \geq 1),$$

$$B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \dots, \quad (1.5)$$

where B_k are the Bernoulli numbers. Finally, we have the well-known functional relation satisfied by $\zeta(s)$ given by [3, p. 603]

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s) \Gamma(1-s) \sin \frac{1}{2} \pi s. \quad (1.6)$$

2. An integral representation

The generalised Mathieu series defined in (1.2) can be written as

$$S_{\mu,\gamma}(a; \lambda) = a^{-\delta} \sum_{n=1}^{\infty} h(n/a), \quad h(x) := \frac{x^\gamma}{(1+x^\lambda)^\mu}, \quad \delta := \lambda\mu - \gamma \quad (2.1)$$

where the parameter $\delta > 1$ for convergence. We employ a Mellin transform approach as discussed in [6, Section 4.1.1]. The Mellin transform of $h(x)$ is $\mathcal{H}(s) = \int_0^\infty x^{s-1} h(x) dx$, where

$$\begin{aligned} \mathcal{H}(s) &= \int_0^\infty \frac{x^{\gamma+s-1}}{(1+x^\lambda)^\mu} dx = \frac{1}{\lambda} \int_0^\infty \frac{\tau^{(\gamma+s)/\lambda-1}}{(1+\tau)^\mu} d\tau \\ &= \frac{\Gamma(\frac{\gamma+s}{\lambda}) \Gamma(\mu - \frac{\gamma+s}{\lambda})}{\lambda \Gamma(\mu)} \end{aligned}$$

in the strip $-\gamma < \Re(s) < \delta$. Using the Mellin inversion theorem (see, for example, [6, p. 118]), we find

$$\sum_{n=1}^{\infty} h(n/a) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{c-\infty i}^{c+\infty i} \mathcal{H}(s) (n/a)^{-s} ds = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \mathcal{H}(s) \zeta(s) a^s ds,$$

where $1 < c < \delta$. The inversion of the order of summation and integration is justified by absolute convergence provided $1 < c < \delta$.

Then, from (2.1), we have [4, 11]

$$S_{\mu,\gamma}(a; \lambda) = \frac{a^{-\delta}}{\lambda \Gamma(\mu)} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \Gamma\left(\frac{\gamma+s}{\lambda}\right) \Gamma\left(\mu - \frac{\gamma+s}{\lambda}\right) \zeta(s) a^s ds, \quad (2.2)$$

where $1 < c < \delta$. From the estimates in (1.4), the integral in (2.2) then defines $S_{\mu,\gamma}(a; \lambda)$ for complex a in the sector $|\arg a| < \pi/\lambda$. The asymptotic expansion of $S_{\mu,\gamma}(a; \lambda)$ for large a and real parameters λ, μ and γ , such that $\delta > 1$, is given in (1.4); see [4, Theorem 3] for complex a and [11] for positive a and unrestricted γ .

We now suppose in the remainder of this paper that $\mu > 0$, with $\lambda (> 0)$ and γ both chosen to be even integers. More specifically, we write

$$\lambda = 2p \quad (p = 1, 2, \dots), \quad \gamma = 2m \quad (m = 0, \pm 1, \pm 2, \dots). \tag{2.3}$$

The integration path in (2.2) lies to the right of the simple pole of $\zeta(s)$ at $s = 1$ and the poles of $\Gamma((\gamma + s)/\lambda)$ at $s = -\gamma - \lambda k$ ($k = 0, 1, 2, \dots$), but to the left of the poles of the second gamma function at $s = \delta + \lambda k$. When $\gamma = 2, 4, \dots$, the poles at $s = -\gamma - \lambda k$ are cancelled by the trivial zeros of $\zeta(s)$ at $s = -2, -4, \dots$. When $\gamma = -2m, m = 0, 1, 2, \dots$, however, there is a finite set of poles of this sequence on the left of the integration path situated in $\Re(s) \geq 0$ with $0 \leq k \leq k^*$, where k^* is the index that satisfies

$$m - k^*p \geq 0, \quad m - (k^* + 1)p < 0. \tag{2.4}$$

The remaining poles of this sequence corresponding to $k > k^*$ are cancelled by the trivial zeros of $\zeta(s)$ with the result that there are again no poles in $\Re(s) < 0$.

We consider the integral in (2.2) taken round the rectangular contour with vertices at $c \pm iT$ and $-c' \pm iT$, where $c' > 0$ and $T > 0$. The contribution from the upper and lower sides of the rectangle $s = \sigma \pm iT, -c' \leq \sigma \leq c$, vanishes as $T \rightarrow \infty$ provided $|\arg a| < \pi/\lambda$, since from (1.4), the modulus of the integrand is controlled by $O(T^{\Omega(\sigma) + \frac{1}{2}\mu - \frac{3}{4}} \log T e^{-\Delta T})$, where $\Delta = \pi/\lambda - |\arg a|$. Evaluation of the residues then yields

$$S_{\mu,\gamma}(a; \lambda) = \frac{\Gamma(\frac{\gamma+1}{\lambda})\Gamma(\mu - \frac{\gamma+1}{\lambda})}{\lambda\Gamma(\mu)a^{\delta-1}} + H_{\mu,\gamma}(a; \lambda) + J(a), \tag{2.5}$$

where the finite algebraic expansion $H_{\mu,\gamma}(a; \lambda)$ (with $\gamma = 2m$) is given by

$$H_{\mu,\gamma}(a; \lambda) = \begin{cases} 0 & (m = 1, 2, \dots) \\ \frac{a^{-\lambda\mu}}{\Gamma(\mu)} \sum_{k=0}^{k^*} \frac{(-)^k \Gamma(\mu + k)}{k! a^{\lambda k}} \zeta(2m - 2kp) & (m = 0, -1, -2, \dots), \end{cases} \tag{2.6}$$

with the index k^* being defined in (2.4), and

$$J(a) = \frac{a^{-\delta}}{\lambda\Gamma(\mu)} \frac{1}{2\pi i} \int_{-c'-\infty i}^{-c'+\infty i} \Gamma\left(\frac{\gamma + s}{\lambda}\right) \Gamma\left(\mu - \frac{\gamma + s}{\lambda}\right) \zeta(s) a^s ds \quad (c' > 0). \tag{2.7}$$

The values of $\zeta(s)$ at $s = 2, 4, \dots$ can be expressed in terms of the Bernoulli numbers, if so desired, by (1.5).

The integrand in $J(a)$ is holomorphic in $\Re(s) < 0$, so that further displacement of the contour to the left can produce no additional terms in the algebraic expansion of $S_{\mu,\gamma}(a; \lambda)$. We shall see in the next section that $J(a)$ possesses an infinite sequence of increasingly exponentially small terms in the large- a limit.

3. The exponentially small expansion of $J(a)$

In the integral (2.7), we make the change of variable $s \rightarrow -s - \gamma$ to find

$$J(a) = \frac{a^{-\lambda\mu}}{\lambda\Gamma(\mu)} \frac{1}{2\pi i} \int_{d-\infty i}^{d+\infty i} \Gamma\left(\frac{-s}{\lambda}\right) \Gamma\left(\mu + \frac{s}{\lambda}\right) \zeta(-s - \gamma) a^{-s} ds, \quad d = c' - \gamma.$$

We now employ (1.6) to convert the zeta function into one with real part greater than unity. With the parameters λ and γ in (2.3), the above integral can then be written in the form

$$J(a) = \frac{(-)^m a^{-\lambda\mu}}{(2\pi)^\gamma \lambda \Gamma(\mu)} \frac{1}{2\pi i} \int_{d-\infty i}^{d+\infty i} G(s) \zeta(1+s+\gamma) (2\pi a)^{-s} \frac{\sin(\frac{1}{2}\pi s)}{\sin(\frac{\pi s}{\lambda})} ds,$$

where

$$G(s) := \frac{\Gamma(s+\gamma+1)\Gamma(\mu+\frac{s}{\lambda})}{\Gamma(1+\frac{s}{\lambda})}. \tag{3.1}$$

Making use of the expansion

$$\frac{\sin(\frac{1}{2}\pi s)}{\sin(\frac{\pi s}{\lambda})} \equiv \frac{\sin(\frac{\pi p s}{\lambda})}{\sin(\frac{\pi s}{\lambda})} = \sum_{r=0}^{p-1} e^{-i\omega_r s}, \quad \omega_r := (p-1-2r)\frac{\pi}{\lambda}, \tag{3.2}$$

we then obtain

$$J(a) = \frac{(-)^m a^{-\lambda\mu}}{(2\pi)^\gamma \lambda \Gamma(\mu)} \sum_{r=0}^{p-1} \mathcal{E}_r(a), \tag{3.3}$$

where

$$\mathcal{E}_r(a) = \frac{1}{2\pi i} \int_{d-\infty i}^{d+\infty i} G(s) \zeta(s+\gamma+1) (2\pi a e^{i\omega_r})^{-s} ds \tag{3.4}$$

and $d + \gamma = c' > 0$.

The integrals $\mathcal{E}_r(a)$ ($0 \leq r \leq p - 1$) have no poles to the right of the integration path, so that we can displace the path as far to the right as we please. On such a displaced path, which we denote by L , $|s|$ is everywhere large. Let M denote an arbitrary positive integer. The quotient of gamma functions

in $G(s)$ may be expanded by appealing to the inverse-factorial expansion given in [6, p. 53] to obtain

$$G(s) = \lambda^{1-\mu} \left\{ \sum_{j=0}^{M-1} (-)^j c_j \Gamma(s+\vartheta-j) + \rho_M(s) \Gamma(s+\vartheta-M) \right\}, \quad \vartheta := \mu + \gamma, \quad (3.5)$$

where $c_0 = 1$ and $\rho_M(s) = O(1)$ as $|s| \rightarrow \infty$ in $|\arg s| < \pi$. An algorithm for the evaluation of the coefficients c_j is discussed in Section 4.

Substitution of the expansion (3.5) into (3.4) then produces

$$\mathcal{E}_r(a) = \lambda^{1-\mu} \left\{ \sum_{j=0}^{M-1} (-)^j c_j \frac{1}{2\pi i} \int_L \Gamma(s+\vartheta-j) \zeta(s+\gamma+1) (2\pi a e^{i\omega_r})^{-s} ds + R_{M,r} \right\}, \quad (3.6)$$

where the remainders $R_{M,r}$ are given by

$$R_{M,r} = \frac{1}{2\pi i} \int_L \rho_M(s) \Gamma(s+\vartheta-M) \zeta(s+\gamma+1) (2\pi a e^{i\omega_r})^{-s} ds. \quad (3.7)$$

The integrals appearing in (3.6) can be evaluated by making use of the well-known result

$$\frac{1}{2\pi i} \int_{L'} \Gamma(s + \alpha) z^{-s} ds = z^\alpha e^{-z} \quad (|\arg z| < \frac{1}{2}\pi),$$

where L' is a path parallel to the imaginary s -axis lying to the right of all the poles of $\Gamma(s + \alpha)$; see, for example, [6, Section 3.3.1]. Upon expansion of the zeta function in (3.6) (since on L its argument satisfies $\Re(s) + \gamma + 1 > 1$) we find

$$\begin{aligned} \frac{1}{2\pi i} \int_L \Gamma(s+\vartheta-j) \zeta(s+\gamma+1) (2\pi a e^{i\omega_r})^{-s} ds &= \sum_{n=1}^{\infty} \frac{(2\pi n a e^{i\omega_r})^{\vartheta-j}}{n^{1+\gamma}} \exp[-2\pi n a e^{i\omega_r}] \\ &= X_r^{\vartheta-j} e^{-X_r} K_j(X_r; \mu), \quad X_r := X e^{i\omega_r}, \quad X := 2\pi a, \end{aligned} \quad (3.8)$$

where we have defined the exponential sum

$$K_j(X_r; \mu) := \sum_{n=1}^{\infty} \frac{e^{-(n-1)X_r}}{n^{1-\mu+j}}. \quad (3.9)$$

This evaluation is valid provided that the variable X_r satisfies the convergence conditions

$$|\arg a + \omega_r| < \frac{1}{2}\pi \quad (0 \leq r \leq p-1).$$

From the definition of ω_r in (3.2), it is easily verified that these conditions are met when $|\arg a| < \pi/\lambda$. It is then evident that $K_j(X_r; \mu) \sim 1$ as $|a| \rightarrow \infty$ in $|\arg a| < \pi/\lambda$.

Thus we find

$$\mathcal{E}_r(a) = \lambda^{1-\mu} e^{-X_r} \sum_{j=0}^{M-1} (-)^j c_j X_r^{\vartheta-j} K_j(X_r; \mu) + R_{M,r}. \tag{3.10}$$

Bounds for the remainders of the type $R_{M,r}$ have been considered in [6, p. 71]; see also [1, §10.1]. The integration path in (3.7) is such that $\Re(s) + \gamma + 1 > 1$, so that we may employ the bound $|\zeta(x + iy)| \leq \zeta(x)$ for real x, y with $x > 1$. A slight modification of Lemma 2.7 in [6, p. 71] then shows that

$$R_{M,r} = O(X_r^{\vartheta-M} e^{-X_r}) \tag{3.11}$$

as $|a| \rightarrow \infty$ in the sector $|\arg a| < \pi/\lambda$.

The expansion of $S_{\mu,\gamma}(a; \lambda)$ then follows from (2.5), (3.3), (3.10) and (3.11) and is given in the following theorem.

Theorem 1. *Let $\mu > 0$, $\gamma = 2m$, $\lambda = 2p$, where $m = 0, \pm 1, \pm 2, \dots$ and $p = 1, 2, \dots$. Further, let M denote a positive integer, $\omega_r = \pi(p - 1 - 2r)/(2p)$ for $0 \leq r \leq p - 1$ and $\delta = \lambda\mu - \gamma$, $\vartheta = \mu + \gamma$. Then*

$$S_{\mu,\gamma}(a; \lambda) = \frac{\Gamma(\frac{\gamma+1}{\lambda})\Gamma(\mu - \frac{\gamma+1}{\lambda})}{\lambda\Gamma(\mu)a^{\delta-1}} + H_{\mu,\gamma}(a; \lambda) + \frac{(-)^m}{\Gamma(\mu)} \left(\frac{\pi}{p}\right)^\mu a^{\mu-\delta} \sum_{r=0}^{p-1} E_r(a) \tag{3.12}$$

as $|a| \rightarrow \infty$ in the sector $|\arg a| < \pi/\lambda$. The finite algebraic expansion $H_{\mu,\gamma}(a; \lambda)$ is defined in (2.6) and the exponentially small expansions $E_r(a)$ are given by

$$E_r(a) = e^{-X_r + i\vartheta\omega_r} \left\{ \sum_{j=0}^{M-1} (-)^j c_j X_r^{-j} K_j(X_r; \mu) + O(X_r^{-M}) \right\} \quad (0 \leq r \leq p - 1), \tag{3.13}$$

where the leading coefficient $c_0 = 1$ and $X_r = 2\pi a e^{i\omega_r}$. The infinite exponential sums $K_j(X_r; \mu)$ are defined in (3.9).

When a is a real variable, the expansion in Theorem 1 can be expressed in a different form. We have the following theorem.

Theorem 2. *Let the parameters μ, γ, λ and $\delta, \vartheta, \omega_r, m, p$ be as in Theorem 1. Then with $N = \lfloor \frac{1}{2}p \rfloor$ and $X = 2\pi a$, the expansion for $S_{\mu,\gamma}(a; \lambda)$ becomes*

$$S_{\mu,\gamma}(a; \lambda) = \frac{\Gamma(\frac{\gamma+1}{\lambda})\Gamma(\mu - \frac{\gamma+1}{\lambda})}{\lambda\Gamma(\mu)a^{\delta-1}} + H_{\mu,\gamma}(a; \lambda) + \frac{(-)^m}{\Gamma(\mu)} \left(\frac{\pi}{p}\right)^\mu a^{\mu-\delta} \left\{ \sum_{r=0}^{N-1} E_r^*(a) + \begin{pmatrix} 0 \\ \frac{1}{2} E_N^*(a) \end{pmatrix} \right\} \begin{cases} p \text{ even} \\ p \text{ odd} \end{cases} \tag{3.14}$$

as $a \rightarrow +\infty$, where for arbitrary positive integer M

$$E_r^*(a) = 2e^{-X \cos \omega_r} \left\{ \sum_{j=0}^{M-1} (-)^j c_j X^{-j} K_j^*(X; \omega_r) + O(X^{-M}) \right\} \quad (0 \leq r \leq N) \tag{3.15}$$

and the infinite exponential sums $K_j^*(X; \omega_r)$ are defined by

$$K_j^*(X; \omega_r) = \sum_{n=1}^{\infty} \frac{e^{-(n-1)X \cos \omega_r}}{n^{1-\mu+j}} \cos [nX \sin \omega_r + (j - \vartheta) \omega_r].$$

When p is odd, the quantity $\omega_N = 0$.

4. The coefficients c_j

We describe an algorithm for the computation of the coefficients c_j that appear in the exponentially small expansions $E_r(a)$ and $E_r^*(a)$ in (3.13) and (3.15). The expression for the ratio of gamma functions in $G(s)$ in (3.5) may be written in the form

$$\frac{G(s)}{\Gamma(s + \vartheta)} = \lambda^{1-\mu} \left\{ \sum_{j=0}^{M-1} \frac{c_j}{(1 - s - \vartheta)_j} + \frac{\rho_M(s)}{(1 - s - \vartheta)_M} \right\},$$

where $(\alpha)_j = \Gamma(\alpha + j)/\Gamma(\alpha)$ is the Pochhammer symbol. If we introduce the scaled gamma function $\Gamma^*(z) = \Gamma(z)/(\sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z})$, then we have

$$\Gamma(\beta s + \gamma) = \Gamma^*(\beta s + \gamma) (2\pi)^{\frac{1}{2}} e^{-\beta s} (\beta s)^{\beta s + \gamma - \frac{1}{2}} \mathbf{e}(\beta s; \gamma),$$

where

$$\mathbf{e}(\beta s; \gamma) := \exp \left[(\beta s + \gamma - \frac{1}{2}) \log \left(1 + \frac{\gamma}{\beta s} \right) - \gamma \right].$$

The above ratio of gamma functions may therefore be expressed as

$$R(s)\Upsilon(s) = \sum_{j=0}^{M-1} \frac{c_j}{(1 - s - \vartheta)_j} + \frac{\rho_M(s)}{(1 - s - \vartheta)_M} \tag{4.1}$$

as $|s| \rightarrow \infty$ in $|\arg s| < \pi$, where

$$R(s) := \frac{\mathbf{e}(s; \gamma + 1) \mathbf{e}(s/\lambda; \mu)}{\mathbf{e}(s/\lambda; 1) \mathbf{e}(s; \vartheta)}, \quad \Upsilon(s) := \frac{\Gamma^*(s + \gamma + 1) \Gamma^*(\mu + s/\lambda)}{\Gamma^*(1 + s/\lambda) \Gamma^*(s + \vartheta)}.$$

We now let $\xi := s^{-1}$ and follow the procedure described in [6, p.47]. We expand $R(s)$ and $\Upsilon(s)$ for $\xi \rightarrow 0$ making use of the well-known expansion (see, for example, [6, p. 71])

$$\Gamma^*(z) \sim \sum_{k=0}^{\infty} (-)^k \gamma_k z^{-k} \quad (|z| \rightarrow \infty; |\arg z| < \pi),$$

where γ_k are the Stirling coefficients, with

$$\gamma_0 = 1, \quad \gamma_1 = -\frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = \frac{139}{51840}, \quad \gamma_4 = -\frac{571}{2488320}, \dots$$

After some straightforward algebra we find that

$$R(s) = 1 + \frac{1}{2}(\mu - 1)\{(\lambda - 1)\mu - 2\gamma\}\xi + O(\xi^2),$$

$$\Upsilon(s) = 1 - \frac{1}{12}(\mu - 1)(\lambda^2 - 1)\xi^2 + O(\xi^3),$$

so that upon equating coefficients of ξ in (4.1) we can obtain c_1 . The higher coefficients can be obtained by matching coefficients recursively with the aid of *Mathematica* to find

$$c_0 = 1, \quad c_1 = \frac{1}{2}(\mu - 1)\{2\gamma - (\lambda - 1)\mu\},$$

$$c_2 = \frac{1}{24}(\mu - 1)(\mu - 2)\{12\gamma(\gamma - (\lambda - 1)\mu - 1) + (\lambda - 1)\mu(5 - 3\mu + \lambda(3\mu - 1))\},$$

$$c_3 = -\frac{1}{48}(\mu - 1)(\mu - 2)(\mu - 3)\{2 - 2\gamma + (\lambda - 1)\mu\}\{4\gamma(\gamma - (\lambda - 1)\mu - 2) + (\lambda - 1)\mu(3 + \lambda(\mu - 1) - \mu)\}, \dots \quad (4.2)$$

The rapidly increasing complexity of the coefficients with $j \geq 4$ prevents their presentation. However, this procedure is found to work well in specific cases when the various parameters have numerical values, where many coefficients have been so calculated. In Table 1 we present some values² of the coefficients c_j for $1 \leq j \leq 10$, which are used in the specific examples considered in Section 5.

4.1 The coefficients c_j when $\lambda = 2, \gamma = 0$

When $\lambda = 2$ and $\gamma = 0$, it is possible to express the coefficients c_j in closed form for arbitrary $\mu > 0$. From (3.1), we have

$$G(s) = \frac{\Gamma(1 + s)\Gamma(\mu + \frac{1}{2}s)}{\Gamma(1 + \frac{1}{2}s)} = \frac{2^s}{\sqrt{\pi}} \Gamma(\frac{1}{2} + \frac{1}{2}s)\Gamma(\mu + \frac{1}{2}s)$$

upon use of the duplication formula for the gamma function. The inverse factorial expansion of a product of two gamma functions with equal coefficients of s is given in [6, pp. 51–52] in the form

$$\Gamma(s + \alpha)\Gamma(s + \beta) \sim 2^{\frac{3}{2}-\alpha-\beta-2s} \sqrt{\pi} \sum_{j=0}^{\infty} d_j \Gamma(2s + \alpha + \beta - \frac{1}{2} - j)$$

as $|s| \rightarrow \infty$ in $|\arg s| < \pi$, where the coefficients satisfy $d_0 = 1$ and

$$d_j = \frac{2^{-j}}{j!} \prod_{r=1}^j \{(\alpha - \beta)^2 - (r - \frac{1}{2})^2\} \quad (j \geq 1).$$

²In the tables we write the values as $x(y)$ instead of $x \times 10^y$.

Table 1: The coefficients c_j ($1 \leq j \leq 10$) for different γ when $\mu = 5/4$ and $\lambda = 4$.

j	$\gamma = 0$	$\gamma = 2$	$\gamma = -2$
1	-4.6875000000(-1)	+3.1250000000(-2)	-2.1250000000(+0)
2	-3.5888671875(-1)	+1.5673828125(-1)	-2.5546875000(+0)
3	-4.0534973145(-1)	+2.3551940918(-1)	-6.8701171875(+0)
4	-3.3581793308(-1)	+1.6646325588(-1)	-2.5683746338(+1)
5	+7.5268601999(-1)	-9.0884858742(-1)	-1.1944799423(+2)
6	+6.4821335676(+0)	-6.6501405553(+0)	-6.6193037868(+2)
7	+2.6358910987(+1)	-2.7627888119(+1)	-4.2794038211(+3)
8	+4.5855530043(+1)	-5.8401959193(+1)	-3.1831413077(+4)
9	-3.7955573596(+2)	+2.8858407940(+2)	-2.6901844936(+5)
10	-5.1286970180(+3)	+4.6231064924(+3)	-2.5504879368(+6)

Putting $\alpha = \frac{1}{2}$ and $\beta = \mu$, with $s \rightarrow \frac{1}{2}s$, we therefore obtain the coefficients in the inverse factorial expansion of $G(s)$ when $\lambda = 2$, $\gamma = 0$ and $\mu > 0$ given by

$$c_j = \frac{(-2)^{-j}}{j!} \prod_{r=1}^j (\mu - r)(\mu + r - 1) \quad (j \geq 1). \tag{4.3}$$

4.2 The coefficients c_j when μ is an integer

A study of the coefficients c_j with the aid of *Mathematica* enables us to conjecture that they possess the general form

$$c_j = (\mu - 1)(\mu - 2) \dots (\mu - j) P_j(\mu, \gamma, \lambda) \quad (j \geq 1),$$

where P_j denotes a polynomial of degree j in the parameters μ , γ and λ . This implies that the sequence of coefficients is finite for integer values of μ ; that is, for positive integer q , we have

$$c_j = 0 \quad (j \geq q; \mu = q, q = 1, 2, \dots).$$

This can also be seen from (3.1) where, with $\mu = q$,

$$G(s) = \frac{\Gamma(s+\gamma+1)\Gamma(q+\frac{s}{\lambda})}{\Gamma(1+\frac{s}{\lambda})} = \Gamma(s+\gamma+1) \left(1 + \frac{s}{\lambda}\right)_{q-1}. \tag{4.4}$$

When $q = 1$, we have $\vartheta = 1 + \gamma$ and the expansion (3.5) is satisfied trivially by terminating the series at the leading term with $\rho_1(s) \equiv 0$. When $q \geq 2$, the

right-hand side of (3.5) must terminate at $M = q$ with $\rho_q(s) \equiv 0$ to yield

$$G(s) = \lambda^{1-q} \Gamma(s + \gamma + 1) \sum_{j=0}^{q-1} (-)^j c_j (s + \gamma + 1)_{q-j-1} \tag{4.5}$$

in order to have the polynomials in s in (4.4) and (4.5) of the same degree.

Then the exponential expansions $E_r(a)$ in (3.13) become the *finite* sums

$$E_r(a) = e^{-X_r + i\vartheta\omega_r} \sum_{j=0}^{q-1} (-)^j c_j X_r^{-j} K_j(X_r; q), \quad \vartheta = q + \gamma,$$

where the infinite sums $K_j(X_r; q)$ may be expressed exactly in terms of derivatives of an exponential by

$$\begin{aligned} K_j(X_r; q) &= e^{X_r} \sum_{n=1}^{\infty} n^{q-j-1} e^{-nX_r} = (-1)^{q-j-1} e^{X_r} D^{q-j-1} \sum_{n=1}^{\infty} e^{-nX_r} \\ &= (-)^{q-j-1} e^{X_r} D^{q-j-1} (e^{X_r} - 1)^{-1} \quad (|\arg a| < \pi/\lambda), \end{aligned} \tag{4.6}$$

with $D \equiv d/dX_r$. The coefficients c_j ($0 \leq j \leq \mu - 1$) are obtained by recursive solution of (4.4) and (4.5) and are given below³ for $\mu = 1, 2, \dots, 5$:

$$\begin{aligned} \mu = 1: & \quad c_0 = 1 \\ \mu = 2: & \quad c_0 = 1, \quad c_1 = \gamma - \lambda - 1 \\ \mu = 3: & \quad c_0 = 1, \quad c_1 = 2\gamma - 3\lambda + 3, \quad c_2 = (1 + \gamma - \lambda)(1 + \gamma - 2\lambda) \\ \mu = 4: & \quad c_0 = 1, \quad c_1 = 3\gamma - 6\lambda + 6, \quad c_2 = 3\gamma^2 + 9\gamma + 7 - 18\lambda - 12\lambda\gamma + 11\lambda^2, \\ & \quad c_3 = (1 + \gamma - \lambda)(1 + \gamma - 2\lambda)(1 + \gamma - 3\lambda) \\ \mu = 5: & \quad c_0 = 1, \quad c_1 = 4\gamma - 10\lambda + 10, \quad c_2 = 6\gamma^2 + 24\gamma + 25 - 60\lambda - 30\lambda\gamma + 35\lambda^2, \\ & \quad c_3 = (2\gamma - 5\lambda + 3)(2\gamma^2 + 6\gamma + 5 - 15\lambda - 10\lambda\gamma + 10\lambda^2), \\ & \quad c_4 = (1 + \gamma - \lambda)(1 + \gamma - 2\lambda)(1 + \gamma - 3\lambda)(1 + \gamma - 4\lambda). \end{aligned} \tag{4.7}$$

The generalised Mathieu series when μ is an integer can therefore be expressed by the following closed-form evaluation.

Theorem 3. *Let the parameters $\gamma, \lambda, \omega_r$ and m, p be as in Theorem 1. Let $\mu = q$, where q is a positive integer, and $\delta = \lambda q - \gamma, \vartheta = q + \gamma$. Then with $X_r = 2\pi a e^{i\omega_r}$, the generalised Mathieu series has the closed-form evaluation*

$$S_{q,\gamma}(a; \lambda) = \frac{\Gamma(\frac{\gamma+1}{\lambda})\Gamma(q - \frac{\gamma+1}{\lambda})}{\lambda\Gamma(q)a^{\delta-1}} + H_{q,\gamma}(a; \lambda) + \frac{(-)^m}{\Gamma(q)} \left(\frac{\pi}{p}\right)^q a^{q-\delta} \sum_{r=0}^{p-1} E_r(a), \tag{4.8}$$

³The values of c_1, c_2 and c_3 follow from (4.2).

where $H_{q,\gamma}(a; \lambda)$ is defined in (2.6) and

$$E_r(a) = e^{-X_r + i\vartheta\omega_r} \sum_{j=0}^{q-1} (-)^j c_j X_r^{-j} K_j(X_r; q) \tag{4.9}$$

with the sums $K_j(X_r; q)$ expressed as derivatives of the exponential term in (4.6). The coefficients c_j ($0 \leq j \leq q - 1$) are given in (4.7) for $q \leq 5$.

5. Numerical results and concluding remarks

We present some examples of the large- a expansion of $S_{\mu,\gamma}(a; \lambda)$ given in Sections 3 and 4 to demonstrate numerically the validity of our results.

Example 1. We select $\lambda = 4$ ($p = 2, N = 1$), so that for $\gamma = 2m$ and $\mu > 0$ we obtain from Theorem 2

$$S_{\mu,\gamma}(a; 4) - \frac{\Gamma(\frac{\gamma+1}{4})\Gamma(\mu - \frac{\gamma+1}{4})}{4\Gamma(\mu) a^{4\mu-\gamma-1}} - H_{\mu,\gamma}(a; 4) = \frac{(-)^m(\pi/2)^\mu}{\Gamma(\mu) a^{3\mu-\gamma}} E_0^*(a), \tag{5.1}$$

as $a \rightarrow +\infty$, where

$$E_0^*(a) = 2e^{-X/\sqrt{2}} \left\{ \sum_{j=0}^{M-1} (-)^j c_j X^{-j} K_j^*(X; \frac{1}{4}\pi) + O(X^{-M}) \right\},$$

and

$$K_j^*(X; \frac{1}{4}\pi) = \sum_{n=1}^{\infty} \frac{e^{-(n-1)X/\sqrt{2}}}{n^{1-\mu+j}} \cos \left[\frac{nX}{\sqrt{2}} + (j - \mu - \gamma) \frac{\pi}{4} \right]$$

with $X = 2\pi a$ and the coefficients $c_j \equiv c_j(\mu, \gamma)$. The leading term on the right-hand side of (5.1) is easily seen to be given by

$$\frac{2(-)^m(\pi/2)^\mu}{\Gamma(\mu) a^{3\mu-\gamma}} e^{-X/\sqrt{2}} \cos \left[\frac{X}{\sqrt{2}} - (\mu + \gamma) \frac{\pi}{4} \right] \quad (a \rightarrow +\infty).$$

This last approximation agrees with that obtained in [10] using different methods.

For numerical comparison, we take $\mu = \frac{5}{4}$ and three values of $\gamma = 0, \pm 2$. Then, we have from (5.1) the expansion

$$\begin{aligned} \hat{S} &:= S_{\frac{5}{4},\gamma}(a; 4) - \frac{\Gamma(\frac{2+1}{4})\Gamma(1 - \frac{2}{4})}{\Gamma(\frac{1}{4}) a^{4-\gamma}} - H_{\frac{5}{4},\gamma}(a; 4) \\ &\sim \frac{(-)^m 2^{7/4} \pi^{5/4}}{\Gamma(\frac{1}{4}) a^{15/4-\gamma}} e^{-X/\sqrt{2}} \sum_{j=0}^{\infty} \frac{(-)^j c_j}{X^j} \sum_{n=1}^{\infty} \frac{e^{-(n-1)X/\sqrt{2}}}{n^{j-1/4}} \cos \left[\frac{nX}{\sqrt{2}} + (j - \frac{5}{4} - \gamma) \frac{\pi}{4} \right], \end{aligned} \tag{5.2}$$

Table 2: The absolute relative error in the computation of \hat{S} from (5.2) for different γ and truncation index j when $\mu = 5/4$, $\lambda = 4$ and $a = 5$.

j	$\gamma = 0$ $\hat{S} = -1.54766(-12)$	$\gamma = 2$ $\hat{S} = -3.59325(-11)$	$\gamma = -2$ $\hat{S} = +5.75174(-14)$
0	1.980(-02)	1.178(-04)	3.237(-04)
1	3.378(-04)	1.568(-04)	1.530(-03)
2	1.102(-07)	1.086(-05)	2.066(-04)
3	3.649(-07)	1.758(-07)	1.920(-05)
4	2.697(-08)	4.874(-09)	3.210(-07)
6	2.785(-11)	1.265(-09)	1.437(-07)
8	1.220(-11)	3.334(-12)	1.363(-09)
10	9.421(-14)	1.497(-12)	9.459(-10)
12	2.607(-14)	1.277(-14)	2.768(-11)

as $a \rightarrow +\infty$, where from (2.6) the algebraic expansion is

$$H_{\frac{5}{4}, \gamma}(a; 4) = 0 \quad (\gamma = 2), \quad -\frac{1}{2a^5} \quad (\gamma = 0), \quad \frac{\pi^2}{6a^5} \quad (\gamma = -2).$$

The coefficients c_j (with $c_0 = 1$) for the above three values of γ are tabulated in Table 1 for $1 \leq j \leq 10$. In Table 2, we show the absolute relative error in the computation of \hat{S} (defined as the left-hand side of (5.2)) for different γ and truncation index j using the expansion (5.2) when $a = 5$. The corresponding value of \hat{S} is indicated at the head of each column. In these computations the sum over n was evaluated to an accuracy commensurate with the overall level of precision. The index $j = 12$ corresponds approximately to optimal truncation of $E_0^*(a)$; that is, truncation at or near the least term in absolute value.

Example 2. In Theorem 3, we first consider the case $\mu = 1$ where, from (4.9),

$$e^{-i\vartheta\omega_r} E_r(a) = e^{-X_r} K_0(X_r; 1) = \frac{e^{-X_r}}{1 - e^{-X_r}} = \frac{1}{2} \coth(\pi a e^{i\omega_r}) - \frac{1}{2}.$$

Some straightforward algebra shows, when $\vartheta = 1 + 2m$, that

$$\sum_{r=0}^{p-1} e^{i\vartheta\omega_r} = e^{\pi i\vartheta(p-1)/(2p)} \sum_{r=0}^{p-1} e^{-\pi i\vartheta r/p} = \frac{\sin(\pi\vartheta/2)}{\sin(\pi\vartheta/\lambda)} = \frac{(-)^m}{\sin(\pi\vartheta/\lambda)}.$$

It then follows from (4.8) (where the first term in (4.8) involving $a^{1-\delta}$ cancels

with the contribution from the above finite sum) that

$$\begin{aligned}
 S_{1,2m}(a; 2p) &= \sum_{n=1}^{\infty} \frac{n^{2m}}{n^{2p} + a^{2p}} \\
 &= H_{1,2m}(a; 2p) + \frac{(-)^m \pi}{2pa^{2p-2m-1}} \sum_{r=0}^{p-1} e^{i\vartheta\omega_r} \coth(\pi a e^{i\omega_r}). \quad (5.3)
 \end{aligned}$$

Series of this type have been expressed as infinite sums of Riemann zeta functions in [7].

In the case $\gamma = 0, \lambda = 2$, we have $m = 0, p = 1, \omega_0 = 0$ and $H_{1,0}(a; 2) = -1/(2a^2)$. Then (5.3) yields the well-known result

$$S_{1,0}(a; 2) = \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth \pi a - \frac{1}{2a^2}.$$

When $\mu = 3, \gamma = \lambda = 2$, we have $\omega_0 = 0$ and from (4.9)

$$E_0(a) = \sum_{j=0}^2 c_j X^{-j} D^{2-j} (e^X - 1)^{-1}, \quad X = 2\pi a.$$

The coefficients for these values of μ, γ and λ are, from (4.7), found to be $c_0 = 1, c_1 = 1$ and $c_2 = -1$, whence

$$S_{3,2}(a; 2) = \sum_{n=1}^{\infty} \frac{n^2}{(n^2 + a^2)^3} = \frac{\pi}{16a^3} - \frac{\pi^3}{2a} \frac{e^{-X}}{1 - e^{-X}} \left\{ \frac{(1 + e^{-X})}{(1 - e^{-X})^2} + \frac{1}{X(1 - e^{-X})} - \frac{1}{X^2} \right\}.$$

Similarly, if $\gamma = 2, \lambda = 4$ we have $\omega_0 = \frac{1}{4}, \omega_1 = -\frac{1}{4}$ and $c_0 = 1, c_1 = -5$ and $c_2 = 5$. Then, for $a > 0$, we obtain

$$\begin{aligned}
 S_{3,2}(a; 4) &= \sum_{n=1}^{\infty} \frac{n^2}{(n^4 + a^4)^3} = \frac{5\pi\sqrt{2}}{128a^9} \\
 &+ \frac{\pi^3}{8a^7} \Re \left\{ \frac{e^{-X_0 + \pi i/4}}{1 - e^{-X_0}} \left[\frac{(1 + e^{-X_0})}{(1 - e^{-X_0})^2} + \frac{5}{X_0(1 - e^{-X_0})} + \frac{5}{X_0^2} \right] \right\}, \quad X_0 = 2\pi a e^{\pi i/4}.
 \end{aligned}$$

Finally, we remark that the asymptotic expansion of the alternating version of (1.2) can be deduced by making use of the identity

$$\tilde{S}_{\mu,\gamma}(a; \lambda) := \sum_{n=1}^{\infty} \frac{(-)^{n-1} n^\gamma}{(n^\lambda + a^\lambda)^\mu} = S_{\mu,\gamma}(a; \lambda) - 2^{1-\delta} S_{\mu,\gamma}(\frac{1}{2}a; \lambda). \quad (5.4)$$

Substitution of the expansion for $S_{\mu,\gamma}(a; \lambda)$ in (3.12) into (5.4) leads to the introduction of the alternating analogues $\tilde{H}_{\mu,\gamma}(a; \lambda)$ and $\tilde{E}_r(a)$ of the algebraic and exponential expansions given by (with k^* defined in (2.4))

$$\tilde{H}_{\mu,\gamma}(a; \lambda) =$$

$$\begin{cases} 0 & (m = 1, 2, \dots) \\ \frac{a^{-\lambda\mu}}{\Gamma(\mu)} \sum_{k=0}^{k^*} \frac{(-)^k \Gamma(\mu + k)}{k! a^{\lambda k}} (1 - 2^{2m+1+\lambda k}) \zeta(2m - 2kp) & (m = 0, -1, -2, \dots) \end{cases} \quad (5.5)$$

and

$$\tilde{E}_r(a) = e^{-X_r/2+i\vartheta\omega_r} \left\{ \sum_{j=0}^{M-1} (-)^j c_j X_r^{-j} \tilde{K}_j(X_r; \mu) + O(X_r^{-M}) \right\}, \quad (5.6)$$

for $0 \leq r \leq p - 1$, where

$$\begin{aligned} \tilde{K}_j(X_r; \mu) &= -e^{X_r/2} \{ e^{-X_r} K_j(X_r; \mu) - 2^{1-\mu+j} e^{-X_r/2} K_j(\frac{1}{2}X_r; \mu) \} \\ &= -e^{X_r/2} \sum_{n=1}^{\infty} \left\{ \frac{e^{-nX_r}}{n^{1-\mu+j}} - \frac{e^{-nX_r/2}}{(\frac{1}{2}n)^{1-\mu+j}} \right\} \\ &= \sum_{n=1}^{\infty} \frac{e^{-(n-1)X_r}}{(n-\frac{1}{2})^{1-\mu+j}}. \end{aligned}$$

We then have the following theorem.

Theorem 4. *Let the parameters μ, γ, λ and the quantities $\delta, \vartheta, \omega_r$ be as in Theorem 1. Then the expansion of the alternating series is*

$$\tilde{S}_{\mu,\gamma}(a; \lambda) = \tilde{H}_{\mu,\gamma}(a; \lambda) + \frac{(-)^{m-1}}{\Gamma(\mu)} \left(\frac{\pi}{p}\right)^\mu a^{\mu-\delta} \sum_{r=0}^{p-1} \tilde{E}_r(a)$$

as $|a| \rightarrow \infty$ in $|\arg a| < \pi/\lambda$, where $\tilde{H}_{\mu,\gamma}(a; \lambda)$ and $\tilde{E}_r(a)$ are defined in (5.5) and (5.6).

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