Mathematica Aeterna, Vol. 6, 2016, no. 2, 197 - 201

Removable Singularities for very weak solutions of A-harmonic Equations with Differential Form

Yuxia Tong

College of Science, North China University of Science and Technology,
Tangshan, Hebei, 063009, China

Linna Cheng

College of Science, North China University of Science and Technology,
Tangshan, Hebei, 063009, China

Jiantao Gu

College of Science, North China University of Science and Technology,
Tangshan, Hebei, 063009, China

Abstract

The removable singularities for very weak solution of A-harmonic equation with differential form is considered based on the higher integrability of very weak solutions.

Mathematics Subject Classification: 35J60, 58J05

Keywords: removable singularity, very weak solution, differential form

1 Introduction

Differential form has important roles in many fields. They can be used to describe various systems of partial differential equations and to express different geometrical structures on manifolds[1−2]. The aim of this present paper is to obtain the removability theorem of a class of elliptic equation with differential form.

In this present paper, we consider the following A-harmonic equation

$$d^*A(x, du) = B(x, du), \tag{1.1}$$

where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^{l+1}(\mathbb{R}^n)$, $B : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ satisfy the conditions

$$\langle A(x,\xi),\xi\rangle \geq \alpha|\xi|^p, \quad |A(x,\xi)| \leq \beta_1|\xi|^{p-1}, \quad |B(x,\xi)| \leq \beta_2|\xi|^{p-1}, \tag{1.2}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $\alpha, \beta_1, \beta_2 > 0$ are constants, $\max\{1, p-1\} \leq r < p < n$.

Definition 1.1 *A differential form $u \in W^{1,r}_{loc}(\Omega, \wedge^{l-1})$ with $\max\{1, p-1\} \leq r < p$ is called a very weak solution of A-harmonic equation (1.1) if u satisfies*

$$\int_\Omega \langle A(x,du), d\varphi\rangle dx = \int_\Omega B(x,du)\varphi dx. \tag{1.3}$$

for any test function $\varphi \in W^{1,\frac{r}{r-p+1}}(\Omega, \bigwedge^{l-1})$.

Before discussing we refer to some notations we shall use. Throughout this paper, Ω will denote an open, connected subset of $\mathbb{R}^n$, and E is a closed set of zero Lebesgue measure in $\mathbb{R}^n$. In order to avoid some technical difficulties related to the imbedding theorem we shall illustrate our approach only for p smaller than the spatial dimension of Ω.

Definition 1.2 [3,4] *A compact set $E \subset \mathbb{R}^n$ is said to have zero r-capacity for $1 < r \leq n$, if for some bounded domain Ω containing E there exists a sequence $\{\varphi_k(x)\}$, $k = 1, 2, ...,$ of functions $\varphi_k(x) \in C_0^\infty(\Omega)$, such that*

(*1*) $0 \leq \varphi_k(x) \leq 1$,

(*2*) each $\varphi_k(x)$ equals to 1 on its own neighborhood of E,

(*3*) $\lim\limits_{k\to\infty} \|\nabla\varphi_k(x)\|_r = 0$,

(*4*) $\lim\limits_{k\to\infty} \varphi_k(x) = 0, \quad \forall x \in \Omega \setminus E$.

A closed set $E \subset \mathbb{R}^n$ has zero r-capacity if every compact subset of E has zero r-capacity.

Notice that for $r = p - \varepsilon$, $0 < \varepsilon < n - 1$, a closed set $E \subset \mathbb{R}^n$ of Hausdorff dimension $\dim_H(E) < \varepsilon$ has zero r-capacity.

Definition 1.3 [3,4] *Let $E \subset \mathbb{R}^n$ be a compact subset of zero Hausdorff measure of n-dimension in $\mathbb{R}^n$. A peak function defined in E is a function $\rho(x) \in C^\infty(\mathbb{R}^n \setminus E)$ for which $\lim\limits_{x\to a} \rho(x) = \infty$, whenever $a \in E$.*

Next is the main results of this present paper.

Theorem 1.4 *Suppose that Ω is a bounded convex domain in $\mathbb{R}^n$, $E \subset \mathbb{R}^n$ be a compact subset of zero Hausdorff measure of n-dimension in $\mathbb{R}^n$. If $u \in W^{1,r}_{loc}\left(\Omega \setminus E, \wedge^{l-1}\right)$ is a very weak solution of (1.1), and the peak function defined in E satisfies $\rho(x) \in W^{1,n}_{loc}(\Omega)$, then u extends to Ω as a very weak solution of (1.1) in the whole domain Ω. In particular, it belongs to $W^{1,p}_{loc}\left(\Omega, \wedge^{l-1}\right)$.*

2 Proof of Theorem 1.4

Our results significantly dependent on the following Lemma.

Lemma 2.1 [5] *Let Ω be a bounded convex domain of $\mathbf{R}^n$. There exists exponents $1 < r_1 = r_1(n,p,\beta_1,\beta_2) < p < r_2 = r_2(n,p,\beta_1,\beta_2) < \infty$ such that if $u \in W^{1,r_1}_{loc}\left(\Omega, \wedge^{l-1}\right)$ is a very weak solution of (1.1), then $u \in W^{1,r_2}_{loc}\left(\Omega, \wedge^{l-1}\right)$. In particular, $u \in W^{1,p}_{loc}\left(\Omega, \wedge^{l-1}\right)$ is a weak solution of (1.1) in the usual sense.*

The above Lemma is the higher integrability of very weak solutions to equation (1.1). With the aid of it, we can give the proof of our main result.

Proof of Theorem 1.4 Let $u \in W^{1,r}_{loc}\left(\Omega \setminus E, \wedge^{l-1}\right)$ be a very weak solution of (1.1). The proof can be logically divided into three parts.

Step 1. First, we prove that $u \in W^{1,r}_{loc}(\Omega, \wedge^{l-1})$. Let $\rho(x)$ be a peak function defined in E, a sequence $\{\rho_k(x)\}$ of Lipschitz functions defined as follows,

$$\rho_k(x) = \begin{cases} 1, & \text{if } \rho(x) \geq k+1; \\ \rho(x) - k, & \text{if } k \leq \rho(x) \leq k+1; \\ 0, & \text{if } \rho(x) \leq k. \end{cases} \tag{2.1}$$

Each of these functions is equal to 1 in its own neighborhood of E. Moreover, $\lim\limits_{k\to\infty} \rho_k(x) = 0$ for all $x \notin E$. Noticing that $d\rho_k$ is supported in $\Omega_k = \{x \in \Omega : k \leq \rho(x) \leq k+1\}$. For fixed $\varphi \in C_0^\infty(\Omega, \wedge^{l-1})$, let

$$\eta_k(x) = [1 - \rho_k(x)]\varphi(x). \tag{2.2}$$

By (2.1), the sequence $\{\eta_k(x)\}$ is supported in $\Omega \setminus E$, and

$$d\eta_k = -\varphi \wedge d\rho_k + [1 - \rho_k(x)]d\varphi. \tag{2.3}$$

Since $u \in W^{1,r}_{loc}(\Omega \setminus E, \wedge^{l-1})$ is a very weak solution of (1.1), the formula of integration by parts holds for any Lipschitz functions sequence $\{\eta_k(x)\}$ supported in $\Omega \setminus E$, i.e.

$$\int_{\Omega\setminus E} (\eta_k \wedge du)dx = -\int_{\Omega\setminus E} (u \wedge d\eta_k)dx, \quad \forall \eta_k \in C_0^\infty(\Omega \setminus E, \wedge^{l-1}). \tag{2.4}$$

For $\eta_k = (1-\rho_k)\varphi$ for all $\varphi \in C_0^\infty(\Omega, \wedge^{l-1})$, then

$$\begin{aligned} &\int_\Omega ((1-\rho_k)\varphi \wedge du)dx \\ = &-\int_\Omega ((1-\rho_k)u \wedge d\varphi)dx + \int_\Omega (\varphi \wedge u \wedge d\rho_k)dx. \end{aligned} \tag{2.5}$$

Since $|d\rho_k| \leq |d\rho|$, $\lim_{k\to\infty} |d\rho_k| = 0$, a.e., then $\int_\Omega (\varphi \wedge u \wedge d\rho_k)dx \to 0$ when $k \to \infty$. By (2.5),

$$\int_\Omega (\varphi \wedge du)dx = -\int_\Omega (u \wedge d\varphi)dx, \quad \forall \varphi \in C_0^\infty(\Omega, \wedge^{l-1}). \tag{2.6}$$

Hence $u \in W^{1,r}_{loc}(\Omega, \wedge^{l-1})$.

Step 2. Next, we need the result $u \in W^{1,p}_{loc}(\Omega, \wedge^{l-1})$. For $u \in W^{1,r}_{loc}(\Omega, \wedge^{l-1})$ we have proved in step 1, then by Lemma 2.1, we have $u \in W^{1,p}_{loc}(\Omega, \wedge^{l-1})$, that is, u is the weak solution of (1.1) in Ω.

Step 3. Finally, we verify that u is really the weak solution of (1.1) in Ω, i.e.

$$\int_\Omega \langle A(x, du), d\eta\rangle dx = \int_\Omega B(x, du)\eta dx, \quad \forall \eta \in C_0^\infty(\Omega, \wedge^{l-1}). \tag{2.7}$$

Since $u \in W^{1,r}_{loc}(\Omega \setminus E, \wedge^{l-1})$ is the very weak solution of (1.1) in $\Omega \setminus E$,

$$\int_{\Omega\setminus E} \langle A(x, du), d\varphi\rangle dx = \int_{\Omega\setminus E} B(x, du)\varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega \setminus E, \wedge^{l-1}). \tag{2.8}$$

Let

$$\varphi_k = (1 - \rho_k)\eta, \ \forall \eta \in C_0^\infty(\Omega, \wedge^{l-1}), \tag{2.9}$$

we shall use φ_k in (2.9) in place of φ in (2.8), then (2.8) becomes

$$\begin{aligned} &\int_\Omega (1 - \rho_k)\langle A(x, du), d\eta\rangle dx \\ = \ &\int_\Omega \langle A(x, du), \eta \wedge d\rho_k\rangle dx + \int_\Omega (1 - \rho_k)B(x, du)\eta dx. \end{aligned} \tag{2.10}$$

Now we estimate the right-hand side of the above inequality. Noticing that $d\rho_k$ is supported in set $\Omega_k = \{x \in \Omega : k \leq \rho(x) \leq k + 1\}$, $|d\rho_k| \leq |d\rho|$. By condition (i), the Hölder inequality,

$$\begin{aligned} &\left|\int_\Omega \langle A(x, du), \eta \wedge d\rho_k\rangle dx\right| \\ \leq \ &\alpha \int_{\Omega_k} |\eta||du|^{p-1}|d\rho_k|dx \\ \leq \ &\alpha\|\eta\|_\infty \left(\int_{\Omega_k} |du|^p dx\right)^{1-\frac{1}{p}} \left(\int_{\Omega_k} |d\rho|^p dx\right)^{\frac{1}{p}} \\ \leq \ &\alpha\|\eta\|_\infty |\Omega_k|^{\frac{1}{p}-\frac{1}{n}} \left(\int_{\Omega_k} |du|^p dx\right)^{1-\frac{1}{p}} \left(\int_{\Omega_k} |d\rho|^n dx\right)^{\frac{1}{n}}. \end{aligned} \tag{2.11}$$

For $\eta \in C_0^\infty(\Omega, \wedge^{l-1})$, $u \in W_{loc}^{1,p}(\Omega, \wedge^{l-1})$, $\rho \in W_{loc}^{1,n}(\Omega)$, and $|\Omega_k| \to 0$ as $k \to \infty$, then we conclude that the integrals in the above inequality converge to zero. Then (2.10) becomes

$$\int_\Omega \langle A(x, du), d\eta \rangle dx = \int_\Omega B(x, du)\eta dx. \tag{2.12}$$

This completes the proof of Theorem 1.4.
□

ACKNOWLEDGEMENTS. The research is supported by Scientific Research Foundation of Hebei United University(z201219) and National Science Foundation of Hebei Province (A2013209278).

References

[1] R.P.Agarwal, S.Ding, C.Nolder. *Inequalities for differential forms.* Springer, New York, NY, USA, 2009.

[2] R.P.Agarwal, Shusen Ding. *Advances in differential forms and the A-harmonic equation.* Mathematical and Computer Modelling, 2003, 37: 1393-1426.

[3] T.Iwaniec, L.Migliaccio, L.Nania, and C.Sbordone. *Integrability and removability results for quasiregular mappings in high dimensions.* Mathematica Scandinavica, 1994,75(2): 263-279.

[4] Zheng Shenzhou. *Removable Singularities of Solutions of A-harmonic Type Equations.* Acta Mathematicae Applicatae Sinica, English Series, 2004,20(1):115-122.

[5] Tong Yuxia, Zheng Shenzhou, Gu Jiantao. *Higher Integrability for Very Weak Solutions of Inhomogeneous A-Harmonic Form Equations.* Journal of Applied Mathematics, Volume 2014, Article ID 308751, http://dx.doi.org/10.1155/2014/308751.

Received: January 3, 2016