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A Generalization of Weak $\mathcal{WT}_2$-Class of Differential Forms and Its Applications

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Abstract

A generalized class of weak $\mathcal{WT}_2$-class of differential forms is defined, a weak reverse Hölder inequality is obtained for this class, and some applications to the regularity theory of weakly (K_1, K_2)-quasiregular mappings, very weak solutions of nonhomogeneous $\mathcal{A}$-harmonic equations, and generalized solutions of Beltrami system with three characteristic matrices are given.

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1 Introduction

In the present paper, we will deal with a generalization of weak $\mathcal{WT}_2$-class of differential forms defined in [1] by H.Y.Gao and Y.Y.Wang, we will obtain a weakly reverse Hölder inequality for this class of differential forms, and then we will give some applications.

We first introduce some symbols and notations used in this paper. Let Ω be a connected open subset of R^n, $n \geq 2$. We use $e_1, e_2, \cdots, e_n$ to denote the standard unit basis of R^n. Let $\bigwedge^\ell = \bigwedge^\ell(\mathrm{R}^n)$ be the linear space of ℓ-covectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell}$ corresponding to all ordered ℓ-tuples $I = (i_1, i_2, \cdots, i_\ell), 1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$, $\ell = 0, 1, \cdots, n$. A differential ℓ-form ω on Ω is a Schwartz distribution on Ω with values in $\bigwedge^\ell = \bigwedge^\ell(\mathrm{R}^n)$. We write $\omega \in \mathcal{D}'\left(\Omega, \bigwedge^\ell\right)$. For $\alpha = \sum \alpha^I e_I \in \bigwedge^\ell$ and $\beta = \sum \beta^I e_I \in \bigwedge^\ell$, the inner product in $\bigwedge^\ell$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all ℓ-tuples $I = (i_1, i_2, \cdots, i_\ell)$. The Hodge star operator $* : \bigwedge^\ell \to \bigwedge^{n-\ell}$ is defined by the rule $*1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and

$$\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle * 1$$

for all $\alpha, \beta \in \bigwedge^\ell$. The norm of $\alpha \in \bigwedge^\ell$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge *\alpha) \in \bigwedge^0 = \mathrm{R}$. The Hodge star is an isometric isomorphism on $\bigwedge^\ell$. It is obvious that $** = (-1)^{\ell(n-\ell)} : \bigwedge^\ell \to \bigwedge^\ell$. We set $*^{-1}\omega = (-1)^{\ell(n-\ell)} * \omega$ for ω a differential form of degree ℓ. The operator $*^{-1}$ is an inverse to $*$ in the sense that $*^{-1}(*\omega) = *(*^{-1}\omega) = \omega$ for $\omega \in \bigwedge^\ell$.

Let $1 \leq p < \infty$. We denote the L^p-norm of a measurable function f over E by

$$\|f\|_p = \|f\|_{p,E} = \left(\int_E |f(x)|^p dx\right)^{1/p}.$$

We write $L^p\left(\Omega, \bigwedge^\ell\right)$ for the ℓ-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_\ell}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_\ell}$ with $\omega_I \in L^p(\Omega, \mathrm{R})$ for all ordered ℓ-tuples I. Thus $L^p\left(\Omega, \bigwedge^\ell\right)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx\right)^{1/p} = \left(\int_\Omega \left(\sum |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}.$$

Similarly, $W^{1,p}\left(\Omega, \bigwedge^\ell\right)$ are those differential ℓ-forms on Ω whose coefficients are in $W^{1,p}(\Omega, \mathrm{R})$. The notations $W^{1,p}_{loc}(\Omega, \mathrm{R})$ and $W^{1,p}_{loc}\left(\Omega, \bigwedge^\ell\right)$ are self-explanatory.

The exterior derivative is denoted by $d : D'\left(\Omega, \bigwedge^\ell\right) \to D'\left(\Omega, \bigwedge^{\ell+1}\right)$ for $\ell = 0, 1, \cdots, n$. Its formal adjoint operator $d^* : D'\left(\Omega, \bigwedge^{\ell+1}\right) \to D'\left(\Omega, \bigwedge^\ell\right)$ is given by $d^* = (-1)^{n\ell+1} * d*$ on $D'\left(\Omega, \bigwedge^{\ell+1}\right)$, $\ell = 0, 1, \cdots, n$. The well-known Poincaré Lemma states that $d \circ d = 0$. It is easy to see that $d^* \circ d^* = 0$ as well.

A differential ℓ-form $\omega \in \mathcal{D}'\left(\Omega, \bigwedge^\ell\right)$ is called simple if there are differential forms $\alpha_1, \cdots, \alpha_\ell$ of degree 1 such that

$$\omega = \alpha_1 \wedge \cdots \wedge \alpha_\ell.$$

A useful property is: if $\alpha \in \bigwedge^k$ and $\beta \in \bigwedge^\ell$, then

$$|\alpha \wedge \beta| \leq |\alpha||\beta|$$

if at least one of the differential forms α, β is simple.

A differential ℓ-form $u \in \mathcal{D}'\left(\Omega, \bigwedge^\ell\right)$ is called a closed form if $du = 0$ in Ω. It is called exact if there exists a differential form $\alpha \in \mathcal{D}'\left(\Omega, \bigwedge^{\ell-1}\right)$ such that $u = d\alpha$. Poincaré Lemma implies that exact forms are closed. Similarly, a differential $(\ell+1)$-form $v \in \mathcal{D}'\left(\Omega, \bigwedge^{\ell+1}\right)$ is called a coclosed form if $d^*v = 0$. It is called coexact if there exists a differential form $\beta \in \mathcal{D}'\left(\Omega, \bigwedge^\ell\right)$ such that $v = d^*\beta$. Balls with radius R are denoted by B_R and $B_{\sigma R}$ is the ball with the same center as B_R and $\text{diam}(B_{\sigma R}) = \sigma \text{diam}(B_R)$. The n-dimensional Lebesgue measure of a set $E \subset \mathrm{R}^n$ is denoted by $|E|$.

The following result can be found in [2]. Let $Q \subset \mathrm{R}^n$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty\left(Q, \bigwedge^\ell\right) \to C^\infty\left(Q, \bigwedge^{\ell-1}\right)$ defined by

$$(K_y\omega)(x;\xi_1,\cdots,\xi_{\ell-1}) = \int_0^1 t^{\ell-1}\omega(tx+y-ty;x-y,\xi_1,\cdots,\xi_{\ell-1})dt$$

and the decomposition

$$\omega = d(K_y) + K_y(d\omega).$$

Another linear operator $T_Q : C^\infty\left(Q, \bigwedge^\ell\right) \to C^\infty\left(Q, \bigwedge^{\ell-1}\right)$ is defined by averaging K_y over all points y in Q with

$$T_Q\omega = \int_Q \varphi(y)K_y\omega dy,$$

where $\varphi \in C_0^\infty(Q)$ is normalized by $\int_Q \varphi(y)dy = 1$. The ℓ-form ω_Q is defined by

$$\omega_Q = |Q|^{-1}\int_Q \omega(y)dy$$

if $\ell = 0$; and $\omega_Q = d(T_Q\omega)$ if $\ell = 1, \cdots, n$ for all $\omega \in L^p\left(Q, \bigwedge^\ell\right)$ and $1 \le p < \infty$.

Franke et al introduced in [3] four classes of differential forms on Riemannian manifolds and showed that some differential expressions connected in a natural way to quasiregular mappings are members in these classes. In [4], some counterparts of theorems of Phragmén-Lindelöf and of Ahlfors were proved for differential forms of $\mathcal{WT}$-classes. Gao and Wang [1] gave a definition of weak $\mathcal{WT}_2$-class of differential forms, and obtained its weak reverse Hölder inequality, an alternative proof for the higher integrability result of weakly $\mathcal{A}$-harmonic tensors due to B.Stroffolini is provided.

Definition 1.1[2]**.** *A differential form $\alpha \in L^p_{loc}\left(\Omega, \bigwedge^\ell\right)$ is called weakly closed, if for each differential form $\beta \in W^{1,q}_{loc}(\Omega, \bigwedge^{\ell+1})$, $\frac{1}{p}+\frac{1}{q} = 1$, $1 \le p, q \le \infty$ with*

$$supp\beta \cap \partial\Omega = \emptyset,\ supp\beta = \overline{\{x \in \Omega : \beta \ne 0\}},$$

we have

$$\int_\Omega \langle \alpha, d^*\beta\rangle dx = 0.$$

Remark 1.1 A closed differential form $\alpha \in L^p_{loc}\left(\Omega, \wedge^\ell\right)$ is weakly closed.

Definition 1.2[3]**.** *A weakly closed differential form $\omega \in L^p_{loc}\left(\Omega, \wedge^\ell\right), 0 \leq \ell \leq n, p > 1$ is said to be of the class $\mathcal{WT}_2$ on Ω if there exists a weakly closed differential form $\theta \in L^q_{loc}\left(\Omega, \wedge^{n-\ell}\right)$, $\frac{1}{p}+\frac{1}{q}=1$, such that almost everywhere on Ω the conditions*

$$\overline{\nu}_1|\omega|^p \leq \langle \omega, *\theta\rangle \tag{1.1}$$

and

$$|\theta| \leq \overline{\nu}_2|\omega|^{p-1} \tag{1.2}$$

are satisfied, with constants $\overline{\nu}_1, \overline{\nu}_2 > 0$.

Now let

$$\omega \in L^r_{loc}\left(\Omega, \bigwedge^\ell\right),\ 0 \leq \ell \leq n,\ \max\{1, p-1\} \leq r < p,\ p > 1 \tag{1.3}$$

be a weakly closed differential form on Ω.

Definition 1.3[1]**.** *A weakly closed differential form ω (1.3) is said to be of the class weak $\mathcal{WT}_2$ on Ω, if there exists a weakly closed differential form*

$$\theta \in L^{\frac{r}{p-1}}_{loc}\left(\Omega, \bigwedge^{n-\ell}\right), \tag{1.4}$$

such that the conditions (1.1) and (1.2) hold almost everywhere on Ω, with constants $\nu_1, \nu_2 > 0$.

Remark 1.2 The word *weak* in Definition 1.3 means that the integrable exponent r of ω can be smaller than the natural one p.

We now give a generalization of weak $\mathcal{WT}_2$ class of differential forms.

Definition 1.4. *An exact differential form ω (1.3) is said to be of the class weak $\overline{\mathcal{WT}}_2$ on Ω, if there exists a weakly closed differential form θ (1.4) such that the conditions*

$$|\omega|^p \leq \gamma_1\langle \omega, *\theta\rangle + \gamma_2|\omega|^{p-\delta} + \gamma_3 \tag{1.5}$$

and

$$|\theta| \leq \gamma_4|\omega|^{p-1} + \gamma_5 \tag{1.6}$$

hold almost everywhere on Ω, with constants $\gamma_i > 0$, $i = 1, \cdots, 5$ and $0 < \delta < p-1$.

At the end of this section we introduce three lemmas which will be used in the sequel.

B.Stroffolini proved in [5] the following useful lemma.

Lemma 1.1. *Suppose that $\omega \in \mathcal{D}'(D, \bigwedge^\ell)$ and $d\omega \in L^p(D, \bigwedge^{\ell+1})$, $\ell = 0, 1, \cdots, n$ and $1 < p < n$. Then $\omega - \omega_D$ is in $L^{np/(n-p)}(D, \bigwedge^\ell)$ and we have the following uniform estimate*

$$\left(\int_\Omega |\omega - \omega_D|^{np/(n-p)} dx\right)^{(n-p)/np} \le C(p, n) \left(\int_D |d\omega|^p dx\right)^{1/p}$$

for D a cube or a ball in R^n.

The following lemma can be found in [2,6,7].

Lemma 1.2. *Let $\omega \in \mathcal{D}'(D, \bigwedge^\ell)$ be such that $d\omega \in L^p(D, \bigwedge^{\ell+1})$, $1 < p < \infty$. Then $\omega - \omega_D$ is in $W^{1,p}(D, \bigwedge^\ell)$ and*

$$\|\omega - \omega_D\|_{W^{1,p}(D)} \le C(n, p) \|d\omega\|_{p,D}$$

for D a cube or a ball in R^n.

The following lemma comes from [8].

Lemma 1.3. *Let $B_R \subset B_{2R} \subset\subset \Omega$ be concentric balls centered at x_0. Suppose $g(x) \in L^r(B_{2R})$, $1 < r < \infty$, $f(x) \in L^t(B_{2R})$, $t > r$. If for every $x \in B_{2R}$, we have the estimate*

$$⨍_{B_R} |g(x)|^r dx \le \theta ⨍_{B_{2R}} |g(x)|^r dx + C \left(⨍_{B_{2R}} |g(x)|^s dx\right)^{\frac{r}{s}} + ⨍_{B_{2R}} |f(x)|^r dx,$$

where $1 \le s < r$, $0 \le \theta < 1$ and $⨍_{B_R} g(x) dx = \frac{1}{|B_R|} \int_{B_R} g(x) dx$ stands for the integral mean. Then there exists an exponent $r' = r'(\theta, r, n, C) > r$ such that $g(x) \in L^{r'}_{loc}(\Omega)$, and

$$\left(⨍_{B_R} |g(x)|^{r'} dx\right)^{\frac{1}{r'}} \le C_1 \left\{ \left(⨍_{B_{2R}} |g(x)|^r dx\right)^{\frac{1}{r}} + \left(⨍_{B_{2R}} |f(x)|^{r'} dx\right)^{\frac{1}{r'}} \right\}$$

holds for some $C_1 = C_1(n, C, r, \theta, R_0)$ independent of the cube B_R, where $R_0 = dist(x_0, \partial\Omega)$.

In the sequel, we denote $C(*, \cdots, *), C_j(*, \cdots, *)$ constants that depend only on the variables $*, \cdots, *$, whose value may change even on the same line.

2 Weak Reverse Hölder Inequality for Weak $\overline{\mathcal{WT}}_2$ Class of Differential Forms

Theorem 2.1. *If $\omega \in L^r_{loc}\left(\Omega, \bigwedge^\ell\right)$, $\max\{1, p-1\} < r < p$, is of the class weak $\overline{\mathcal{WT}}_2$, then there exists positive constants $\varepsilon_0 = \varepsilon_0(n, p, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$ and $C_j = C_j(n, p, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$, $j = 1, 2$, such that the following weak reserve Hölder inequality*

$$⨍_{B_R} |\omega|^r dx \le \theta ⨍_{B_{2R}} |\omega|^r dx + C_1 \left(⨍_{B_{2R}} |\omega|^s dx\right)^{\frac{r}{s}} + C_2 \qquad (2.1)$$

holds with $s < r$ *and* $0 < \theta < 1$*, provided that* $|p-r| < \varepsilon_0$.

Proof Let $B_R \subset B_{2R} \subset\subset \Omega$ be concentric balls and $\eta(x) \in C_0^\infty(B_{2R})$ be a cutoff function with $0 \le \eta \le 1$, $\eta \equiv 1$ on B_R and $|\nabla\eta| \le \frac{C(n)}{R}$. Since ω is exact, then there exists a differential form $u \in W^{1,r}_{loc}\left(\Omega, \bigwedge^{\ell-1}\right)$, such that $\omega = du$. Consider the Hodge decomposition (see [6,7])

$$|d(\eta u)|^{r-p} d(\eta u) = d\alpha + h, \tag{2.2}$$

where $d\alpha, h \in L^{r/(r-p+1)}\left(B_{2R}, \bigwedge^\ell\right)$. The following estimate holds

$$\|h\|_{r/(r-p+1)} \le C(n)|p-r| \|d(\eta u)\|_r^{r-p+1}. \tag{2.3}$$

Let

$$E = |d(\eta u)|^{r-p} d(\eta u) - |\eta du|^{r-p} \eta du. \tag{2.4}$$

By an elementary inequality (see [9])

$$\left| |X|^{-\varepsilon} X - |Y|^{-\varepsilon} Y \right| \le \frac{2^\varepsilon(1+\varepsilon)}{1-\varepsilon} |X-Y|^{1-\varepsilon},\ 0 \le \varepsilon < 1,\ X, Y \in R^n,$$

we obtain

$$|E| \le \frac{2^{p-r}(p-r+1)}{r-p+1} |u d\eta|^{r-p+1}. \tag{2.5}$$

By the weak closedness of θ, one has

$$\begin{aligned} &\int_{B_{2R}} \langle d\alpha, *\theta\rangle dx = \int_{B_{2R}} \langle *d\alpha, **\theta\rangle dx \\ =\ & (-1)^{\ell(n-\ell)} \int_{B_{2R}} \langle *d\alpha, \theta\rangle dx = (-1)^{\ell-1} \int_{B_{2R}} \langle d^*(*\alpha), \theta\rangle dx = 0. \end{aligned} \tag{2.6}$$

for all $\alpha \in W^{1,r}\left(B_{2R}, \bigwedge^\ell\right)$. Combining (2.2) with (2.4) and (2.6) yields

$$\int_{B_{2R}} \langle |\eta du|^{r-p} \eta du, *\theta\rangle dx = -\int_{B_{2R}} \langle E, *\theta\rangle dx + \int_{B_{2R}} \langle h, *\theta\rangle dx. \tag{2.7}$$

Now let $B_1 = \{x \in B_{2R} : |du| > 1\}$ and $B_2 = \{x \in B_{2R} : |du| \le 1\}$. It is obvious that $B_1 \bigcap B_2 = \phi$ and $B_{2R} = B_1 \bigcup B_2$. (1.5) and (2.7) imply

$$\begin{aligned} & \int_{B_R} |du|^r dx \le \int_{B_{2R}} \eta^{r-p+1} |du|^r dx \\ =\ & \int_{B_1} \eta^{r-p+1} |du|^r dx + \int_{B_2} \eta^{r-p+1} |du|^r dx \\ \le\ & \int_{B_1} \eta^{r-p+1} |du|^{r-p} |du|^p dx + |B_{2R}| \\ \le\ & \gamma_1 \int_{B_1} \eta^{r-p+1} |du|^{r-p} \langle du, *\theta\rangle dx + \gamma_2 \int_{B_1} |du|^{r-\delta} dx + \gamma_3 \int_{B_1} |du|^{r-p} dx + |B_{2R}| \\ \le\ & \gamma_1 \int_{B_1} \langle |\eta du|^{r-p} \eta du, *\theta\rangle dx + \gamma_2 \int_{B_1} |du|^{r-\delta} dx + (\gamma_3+1)|B_{2R}| \\ \le\ & -\gamma_1 \int_{B_1} \langle E, *\theta\rangle dx + \gamma_1 \int_{B_1} \langle h, *\theta\rangle dx + \gamma_2 \int_{B_1} |du|^{r-\delta} dx + (\gamma_3+1)|B_{2R}| \\ :=\ & \gamma_1 I_1 + \gamma_1 I_2 + \gamma_2 I_3 + (\gamma_3+1)|B_{2R}|, \end{aligned} \tag{2.8}$$

where we have used the facts $|du| \le 1$ on B_2 and $|du| > 1$ on B_1, which imply

$$\int_{B_2} \eta^{r-p+1}|du|^r dx \le |B_{2R}| \quad \text{and} \quad \int_{B_1} |du|^{r-p} dx \le |B_{2R}|,$$

respectively. We now estimate $|I_j|$, $j = 1, 2, 3$. With (1.6), (2.5) and Hölder's inequality, we obtain

$$\begin{aligned} |I_1| &= \left| -\int_{B_1} \langle E, *\theta\rangle dx \right| \le \int_{B_{2R}} |E||\theta| dx \\ &\le \frac{2^{p-r}(p-r+1)}{r-p+1} \int_{B_{2R}} \left[\gamma_4 |du|^{p-1} + \gamma_5\right] |ud\eta|^{r-p+1} dx \\ &\le \frac{C(n)}{R^{r-p+1}} \cdot \frac{2^{p-r}(p-r+1)}{r-p+1} \left[\gamma_4 \int_{B_{2R}} |du|^{p-1}|u|^{r-p+1} dx + \gamma_5 \int_{B_{2R}} |u|^{r-p+1} dx\right]. \end{aligned} \tag{2.9}$$

Notice that $\omega = du$ is not affected when an exact form is added to u, thus we may assume that $u_{B_{2R}}$ is equal to zero. This justifies the application of Lemma 1.1 to the first integral in the right-hand side of (2.9). Hence

$$\begin{aligned} & \frac{C(n)\gamma_4}{R^{r-p+1}} \cdot \frac{2^{p-r}(p-r+1)}{r-p+1} \int_{B_{2R}} |du|^{p-1}|u|^{r-p+1} dx \\ \le\ & \frac{C(n)\gamma_4}{R^{r-p+1}} \cdot \frac{2^{p-r}(p-r+1)}{r-p+1} \left(\int_{B_{2R}} |du|^r dx\right)^{\frac{p-1}{r}} \left(\int_{B_{2R}} |u|^r dx\right)^{\frac{r-p+1}{r}} \\ \le\ & \frac{C(n)\gamma_4}{R^{r-p+1}} \cdot \frac{2^{p-r}(p-r+1)}{r-p+1} \left(\int_{B_{2R}} |du|^r dx\right)^{\frac{p-1}{r}} \left(\int_{B_{2R}} |du|^{\frac{nr}{n+r}} dx\right)^{\frac{(n+r)(r-p+1)}{nr}} \\ \le\ & C(n,\gamma_4)\sigma \int_{B_{2R}} |du|^r dx + \frac{C(n,\gamma_4,\sigma)}{R^r} \left(\int_{B_{2R}} |du|^{\frac{nr}{n+r}} dx\right)^{\frac{n+r}{n}}, \end{aligned} \tag{2.10}$$

where $\sigma > 0$ will be chosen later.

Take $1 < p' < \infty$ such that $\frac{n(r-p+1)p'}{n+(r-p+1)p'} < r$, then using Hölder's inequality and Lemma 1.1 again, we arrive at

$$\begin{aligned} & \int_{B_{2R}} |u|^{r-p+1} dx \le \left(\int_{B_{2R}} |u|^{(r-p+1)p'} dx\right)^{\frac{1}{p'}} \left(\int_{B_{2R}} dx\right)^{\frac{p'-1}{p'}} \\ \le\ & C(n,r)|B_{2R}|^{\frac{p'-1}{p'}} \left(\int_{B_{2R}} |du|^{\frac{n(r-p+1)p'}{n+(r-p+1)p'}} dx\right)^{\frac{n+(r-p+1)p'}{np'}} \end{aligned} \tag{2.11}$$

With (1.6), Hölder's inequality and (2.3), we obtain

$$\begin{aligned} |I_2| &= \left|\int_{B_1} \langle h, *\theta\rangle dx\right| \le \int_{B_{2R}} |h||\theta| dx \le \int_{B_{2R}} \left[\gamma_4|du|^{p-1} + \gamma_5\right] |h| dx \\ &\le \left\|\gamma_4 |du|^{p-1} + \gamma_5\right\|_{\frac{r}{p-1}} \|h\|_{\frac{r}{r-p+1}} \\ &\le C(n)|p-r| \left\|\gamma_4|du|^{p-1} + \gamma_5\right\|_{\frac{r}{p-1}} \|d(\eta u)\|_r^{r-p+1}. \end{aligned}$$

With Lemma 1.2, we obtain

$$\begin{aligned}&||d(\eta u)||_r^{r-p+1}=||\eta du+ud\eta||_r^{r-p+1}\le(||\eta du||_r+||ud\eta||_r)^{r-p+1}\\&\le\left(\frac{C(n)}{R}||u||_r+||du||_r\right)^{r-p+1}\le\left(\frac{C(n)}{R}R||du||_r+||du||_r\right)^{r-p+1}\\&=C(n,p)||du||_r^{r-p+1}.\end{aligned}$$

Therefore

$$\begin{aligned}|I_2| &\le C(n,p)|p-r|\left\|\gamma_4|du|^{p-1}+\gamma_5\right\|_{\frac{r}{p-1}}\|du\|_r^{r-p+1}\\&\le C(n,p,\sigma)|p-r|\left\|\gamma_4|du|^{p-1}+\gamma_5\right\|_{\frac{r}{p-1}}^{\frac{r}{p-1}}+\sigma\|du\|_r^r\\&\le C(n,p,\sigma)|p-r|\gamma_4\int_{B_{2R}}|du|^r dx+C(n,p,\sigma)|p-r|\gamma_5|B_{2R}|.\end{aligned}\tag{2.12}$$

where we have used Young's inequality.

We can assume that $p-\delta<r<p$ since Theorem 2.1 holds with $|p-r|$ sufficiently small. Using Hölder's inequality and Young's inequality, $|I_3|$ can be estimated as

$$\begin{aligned}|I_3| &= \left|\int_{B_1}|du|^{r-\delta}dx\right|\\&\le\left(\int_{B_{2R}}|du|^r dx\right)^{\frac{p-\delta}{r}}\left(\int_{B_{2R}}dx\right)^{\frac{r-p+\delta}{r}}\\&\le\varepsilon\int_{B_{2R}}|du|^r dx+C(\varepsilon)|B_{2R}|.\end{aligned}\tag{2.13}$$

Combining (2.8)-(2.13) and dividing both sides by $|B_R|=\omega_n R^n$, we arrive at

$$\begin{aligned}⨍_{B_R}|du|^r dx &\le[C(n,\gamma_1,\gamma_4)\sigma+C(n,p,\sigma)|p-r|\gamma_4+\sigma+\varepsilon]⨍_{B_{2R}}|du|^r dx\\&+C(n,\gamma_1,\gamma_4,\sigma)\left(⨍_{B_{2R}}|du|^{\frac{nr}{n+r}}dx\right)^{\frac{n+r}{n}}\\&+C(n,\gamma_1,\gamma_5)\frac{2^{p-r}(p-r+1)}{r-p+1}\left(⨍_{B_{2R}}|du|^{\frac{n(r-p+1)p'}{n+(r-p+1)p'}}\right)^{\frac{n+(r-p+1)p'}{np'}}\\&+C(n,p,\sigma)|p-r|\gamma_5+\gamma_3+1+C(\varepsilon).\end{aligned}$$

Take σ and ε sufficiently small, and then r sufficiently close to p, such that $\theta=C(n,\gamma_1,\gamma_4)\sigma+C(n,p,\sigma)|p-r|\gamma_4+\sigma+\varepsilon<1$. Let $s=\max\left\{\frac{nr}{n+r},\frac{n(r-p+1)p'}{n+(r-p+1)p'}\right\}<r$, we arrive at

$$⨍_{B_R}|du|^r dx\le\theta⨍_{B_{2R}}|du|^r dx+C_1\left(⨍_{B_{2R}}|du|^s dx\right)^{\frac{r}{s}}+C_2,$$

where we have used the fact that $t\mapsto\left(⨍_{B_{2R}}|du|^t dx\right)^{\frac{1}{t}}$ is increasing. This completes the proof Theorem 2.1.

3 Some Applications

In this section, we give some applications of the reverse Hölder inequality for weak $\overline{\mathcal{WT}}_2$ classes of differential forms to regularity theory of weakly (K_1, K_2)-quasiregular mappings, very weak solutions of nonhomogeneous $\mathcal{A}$-harmonic equations, and generalized solutions of Beltrami system with three characteristic matrices.

3.1 Weakly (K_1, K_2)-Quasiregular Mappings

We recall the definition for weakly (K_1, K_2)-quasiregular mappings, see [7]. Let $f = (f^1, f^2, \cdots, f^n) \in W^{1,r}_{loc}(\Omega, \mathrm{R}^n)$, $1 \leq r < \infty$. Then f is said to be weakly (K_1, K_2)-quasiregular, $K_1 > 0$, $K_2 \geq 0$, if

$$|Df(x)|^n \leq K_1 n^{n/2} J_f(x) + K_2 \tag{3.1}$$

is satisfied almost everywhere in Ω, here $Df(x)$ denotes the formal derivative of f at x, i.e., the $n \times n$ matrix $\left(\frac{\partial f^i}{\partial x_j}\right)_{1 \leq i,j \leq n}$ of partial derivatives of the coordinate functions f^i of f. Further, $|Df(x)|$ is the Hilbert-Schmidt norm of $Df(x)$:

$$|Df(x)| = (\mathrm{Tr} D^t f(x) Df(x))^{1/2} = \left(\sum_{i,j=1}^{n} \left(\frac{\partial f^i}{\partial x_j} \right)^2 \right)^{1/2}.$$

$J_f(x) = \det Df(x)$ is the Jacobian determinant of f at x.

Quasiregular mappings were first studied by Reshetnyak in 1966-1969. He proved that spatial quasiregular mappings (he used the phrase mappings with bounded distortion for quasiregular mappings) share the fundamental topological properties of complex-analytic functions: non-constant quasiregular mappings are discrete, open and sense-preserving. See [10]. The word quasiregular was introduced in this meaning in 1969 by Martio et al. [11], which found another approach, i.e., modulus of a family of curves (the extremal length) and capacities of condensers, to quasiregular mappings [12,13]. Their work showed the power of the method of the modulus of a family of curves, made these mappings more widely known, and led to a series of important results. Some notable developments in the theory of spatial weakly quasiregular mappings, including the regularity theory, were obtained by Iwaniec and Martin [14,15], by the technique developed by Donalson and Sullivan [16]. For other developments related to quasiregular mappings and (K_1, K_2)-quasiregular mappings, see [6,7,17-21].

The reverse Hölder inequality for weakly (K_1, K_2)-quasiregular mappings has been established in [6]. In this subsection, we will give an alternative proof by using the concept of weak $\overline{\mathcal{WT}}_2$ class of differential forms.

Theorem 3.1. *Let $f = (f^1, f^2, \cdots, f^n) \in W^{1,r}_{loc}(\Omega, R^n)$ be a weakly (K_1, K_2)-quasiregular mapping. Then for any ℓ-tuple $I = (i_1, \cdots, i_\ell)$, $1 \le i_1 < \cdots < i_\ell \le n$, the differential form $\omega = df^I = df^{i_1} \wedge \cdots df^{i_\ell}$ is of the class weak $\overline{\mathcal{WT}}_2$ with $p = \frac{n}{\ell}$.*

Proof It is easy to see that

$$\omega = df^{i_1} \wedge \cdots \wedge df^{i_\ell} = d(f^{i_1} df^{i_2} \wedge \cdots \wedge df^{i_\ell})$$

is exact. Let

$$\theta = (-1)^{\ell(n-\ell)} \sigma(I, J) df^J = (-1)^{\ell(n-\ell)} \sigma(I, J) df^{j_1} \wedge \cdots \wedge df^{j_{n-\ell}},$$

where the complementary $(n-\ell)$-tuple $J = (j_1, \cdots, j_{n-\ell})$, $1 \le j_1 < \cdots < j_\ell \le n$, is obtained from the index $(1, 2, \cdots, n)$ by simply deleting the elements of I, and $\sigma(I, J) = \pm 1$ is defined by

$$\sigma(I, J) df^I \wedge df^J = df^1 \wedge \cdots \wedge df^n = J_f(x) dx.$$

It is obvious that θ is closed, thus it is weakly closed.

Because the differential form ω is simple we obtain by the inequality between the geometric and arithmetic means

$$|\omega|^{1/\ell} \le \left(\prod_{m=1}^{\ell} |df^{i_m}| \right)^{1/\ell} \le \frac{1}{\ell} \sum_{m=1}^{\ell} |df^{i_m}| \le \left(\frac{1}{\ell} \sum_{m=1}^{k} |df^{i_m}|^2 \right)^{1/2}. \quad (3.2)$$

Similarly,

$$|\theta|^{1/(n-\ell)} \le \left(\frac{1}{n-\ell} \sum_{m=1}^{n-\ell} |df^{i_m}|^2 \right)^{1/2}. \quad (3.3)$$

Combining (3.2) and (3.3), and using (3.1) we have

$$\left(\ell |\omega|^{2/\ell} + (n-\ell) |\theta|^{2/(n-\ell)} \right)^{n/2} = |Df(x)|^n \le K_1 n^{n/2} J_f(x) + K_2. \quad (3.4)$$

It is obvious that

$$\langle \omega, *\theta \rangle dx = \omega \wedge **\theta = \sigma(I, J) df^{i_1} \wedge \cdots df^{i_\ell} \wedge df^{j_1} \wedge \cdots df^{j_{n-\ell}} = J_f(x) dx. \quad (3.5)$$

(3.4) and (3.5) imply

$$\ell^{n/2} |\omega|^{n/\ell} \le \left(\ell |\omega|^{2/\ell} + (n-\ell) |\theta|^{2/(n-\ell)} \right)^{n/2} \le K_1 n^{n/2} \langle \omega, *\theta \rangle + K_2.$$

Thus

$$|\omega|^{n/\ell} \le K_1 \left(\frac{n}{\ell} \right)^{n/2} \langle \omega, *\theta \rangle + \frac{K_2}{\ell^{n/2}} \le K_1 \left(\frac{n}{\ell} \right)^{n/2} \langle \omega, *\theta \rangle + \frac{K_2}{n^{n/2}}.$$

Since both ω and θ are simple, then by (3.4), (3.5) and Young's inequality,

$$(n-\ell)^{n/2}|\theta|^{n/(n-\ell)} \le \left(\ell|\omega|^{2/\ell} + (n-\ell)|\theta|^{2/(n-\ell)}\right)^{n/2}$$
$$\le K_1 n^{n/2}|\omega||\theta| + K_2 \le K_1 n^{n/2}\left(C(\varepsilon)|\omega|^{n/\ell} + \varepsilon|\theta|^{n/(n-\ell)}\right) + K_2$$

Take ε small enough such that $K_1 n^{n/2}\varepsilon = (n-\ell)^{n/2}/2$, and divided $(n-\ell)^{n/2}/2$ in both sides of the above inequality yields

$$|\theta| \le \gamma_4|\omega|^{(n-\ell)/\ell} + \gamma_5.$$

This ends the proof of Theorem 3.1.

Corollary 3.1. *Let $f = (f^1, f^2, \cdots, f^n) \in W^{1,r}_{loc}(\Omega, R^n)$, $\max\{1, p-1\} \le r < p$, be a weakly (K_1, K_2)-quasiregular mapping, there exists $p_1 = p_1(n, K_1, K_2)$ and $p_2 = p_2(n, K_1, K_2)$, $p_1 < n < p_2$, such that every weakly (K_1, K_2)-quasiregular mapping $f \in W^{1,p_1}_{loc}(\Omega, R^n)$ actually belongs to $W^{1,p_2}_{loc}(\Omega, R^n)$, that is, f is a (K_1, K_2)-quasiregular mapping in the usual sense.*

Proof. This corollary is a direct consequence of Theorem 2.1, Theorem 3.1 and Lemma 1.3.

3.2 Weakly $\mathcal{A}$-Harmonic Tensors

Let us now consider the nonhomogeneous $\mathcal{A}$-harmonic equation

$$d^*\mathcal{A}(x, du(x)) = d^*\mathcal{B}(x, du), \tag{3.6}$$

where $\mathcal{A}, \mathcal{B} : \Omega \times \bigwedge^\ell(\mathrm{R}^n) \to \bigwedge^\ell(\mathrm{R}^n)$ satisfy the conditions

(i) $|\mathcal{A}(x,\xi)| \le a|\xi|^{p-1}$,

(ii) $\langle \mathcal{A}(x,\xi), \xi\rangle \ge |\xi|^p$,

(iii) $|\mathcal{B}(x,\xi)| \le a|\xi|^{p-\delta}$

for almost every $x \in \Omega$ and all $\xi \in \bigwedge^\ell(\mathrm{R}^n)$. Here $p = \frac{n}{\ell}$, $a \ge 1$, $0 < \delta < p-1$ are constants, and $1 < p < \infty$ is a fixed exponent associated with (3.6).

Definition 3.2. A very weak solution to (3.6) is an element u of the Sobolev space $W^{1,r}_{loc}(\Omega, \bigwedge^{\ell-1})$, $\max\{1, p-1\} \le r < p$, such that

$$\int_\Omega \langle \mathcal{A}(x, du), d\varphi\rangle dx = \int_\Omega \langle \mathcal{B}(x, du), d\varphi\rangle dx$$

for all $\varphi \in W^{1,\frac{r}{r-p+1}}\left(\Omega, \bigwedge^{\ell-1}\right)$ with compact support.

Very weak solutions to $\mathcal{A}$-harmonic equation (3.6) have been the subject of intensive research, see [6,7]. The weak reverse Hölder inequality for very weak solutions to (3.6) has been obtained in [7]. In this subsection, we will give an alternative proof of this result by using the concept of weak $\overline{\mathcal{WT}}_2$ class of differential forms.

Theorem 3.2. *If $u \in W^{1,r}_{loc}(\Omega, \bigwedge^{\ell-1})$, $\max\{1, p-1\} \le r < p$, be a very weak solution to (3.6) with conditions (i), (ii) and (iii), then $\omega = du$ is of the class weak $\overline{\mathcal{WT}}_2$.*

Proof. It is obvious that the differential form $\omega = du \in L^r_{loc}\left(\Omega, \bigwedge^\ell\right)$ is exact. Let $\theta = *^{-1}[\mathcal{A}(x, du) - \mathcal{B}(x, du)] \in L^{\frac{r}{p-1}}_{loc}\left(\Omega, \bigwedge^{n-\ell}\right)$. The weak closedness of θ follows from

$$\begin{aligned} &(-1)^{n+n\ell+1}\int_\Omega \langle \theta, d^*\psi\rangle dx = \int_\Omega \langle \theta, *d*\psi\rangle dx \\ = &\int_\Omega \langle *\theta, d*\psi\rangle dx = \int_\Omega \langle \mathcal{A}(x, du) - \mathcal{B}(x, du), d*\varphi\rangle dx = 0 \end{aligned}$$

for all $\psi = *^{-1}\varphi \in W^{1,\frac{r}{r-p+1}}(\Omega, \bigwedge^{n-\ell+1})$ with $\mathrm{supp}\varphi \cap \partial\Omega = \emptyset$. By (i), (iii) and Young's inequality,

$$\begin{aligned} |\theta| &= |\mathcal{A}(x, du) - \mathcal{B}(x, du)| \le |\mathcal{A}(x, du)| + |\mathcal{B}(x, du)| \\ &\le a|du|^{p-1} + a|du|^{p-1-\delta} \le a\left(|du|^{p-1} + \varepsilon|du|^{p-1} + C(\varepsilon)\right) \\ &= a(1+\varepsilon)|du|^{p-1} + aC(\varepsilon). \end{aligned} \tag{3.7}$$

Further, by (i), (ii) and (iii) we get

$$\begin{aligned} &\langle \omega, *\theta\rangle = \langle du, \mathcal{A}(x, du) - \mathcal{B}(x, du)\rangle \\ = &\ \langle du, \mathcal{A}(x, du)\rangle - \langle du, \mathcal{B}(x, du)\rangle \\ \ge &\ |du|^p - a|du|^{p-\delta}. \end{aligned}$$

This implies

$$|du|^p \le \langle du, *\theta\rangle + a|du|^{p-\delta},$$

completing the proof of Theorem 3.2.

Corollary 3.2. *If $u \in W^{1,r}_{loc}(\Omega, \bigwedge^{\ell-1})$, $\max\{1, p-1\} \le r < p$, be a very weak solution to (3.6) with conditions (i), (ii) and (iii), then there exists $p_1 = p_1(n, a)$ and $p_2 = p_2(n, K_1, K_2)$, $p_1 < p < p_2$, such that every very weak solution $u \in W^{1,p_1}_{loc}(\Omega, \bigwedge^{\ell-1})$ to (3.6) actually belongs to $W^{1,p_2}_{loc}(\Omega, \bigwedge^{\ell-1})$, that is, u is a weak solution to (3.6) in the usual sense.*

Proof. This corollary is a direct consequence of Theorem 2.1, Theorem 3.2 and Lemma 1.3.

3.3 Generalized Solutions to Beltrami System with Three Characteristic matrices

We now consider the Beltrami system with three characteristic matrices

$$D^t f(x)H(x)Df(x) = J_f^{2/n}(x)G(x) + F(x), \tag{3.8}$$

where $H(x), G(x), F(x)$ are positive definite, symmetric $n \times n$ matrices with the following conditions

(I) $\alpha_1|\xi|^2 \le \langle H(x)\xi,\xi\rangle \le \beta_1|\xi|^2$,
(II) $\alpha_2|\eta|^2 \le \langle G(x)\eta,\eta\rangle \le \beta_2|\eta|^2$,
(III) $\alpha_3|\zeta|^2 \le \langle H(x)\zeta,\zeta\rangle \le \beta_3|\zeta|^2$
for all $\xi,\eta,\zeta \in \mathrm{R}^n$ and some positive constants α_i,β_i, $i=1,2,3$.

The Beltrami system with one, two or three characteristic matrices have been studied by many authors, see [6,7,22,23]. In this subsection, we will give a regularity result for generalized solutions to (3.8).

Theorem 3.3. *Let $f=(f^1,f^2,\cdots,f^n)\in W^{1,r}_{loc}(\Omega,R^n)$ be a generalized solution of the Beltrami system (3.8), then $\omega = df^I = df^{i_1}\wedge df^{i_2}\wedge\cdots\wedge df^{i_\ell}$, $1\le \ell \le n$ is of the class weak $\overline{\mathcal{WT}}_2$.*

Proof. We prove Theorem 3.3 by showing that any generalized solution $f=(f^1,f^2,\cdots,f^n)\in W^{1,r}_{loc}(\Omega,\mathrm{R}^n)$ of (3.8) be a weakly (K_1,K_2)-quasiregular mapping, thus Theorem 3.3 follows from Theorem 3.1. In fact, (I), (II), (III) and (3.8) imply

$$\begin{aligned} &\alpha_1|Df|^2 \le \langle H(x)Df(x),Df(x)\rangle = \langle D^t f(x)H(x)Df(x), Id\rangle \\ = \;& J_f^{2/n}(x)\langle G(x),Id\rangle + \langle F(x),Id\rangle \le \beta_2 J_f^{2/n}(x)+\beta_3. \end{aligned}$$

Therefore

$$|Df(x)|^n \le 2^{(n-2)/2}\left[\left(\frac{\beta_2}{\alpha_1}\right)^{n/2} J_f(x) + \left(\frac{\beta_3}{\alpha_1}\right)^{n/2}\right].$$

Corollary 3.3. *Let $f=(f^1,f^2,\cdots,f^n)\in W^{1,r}_{loc}(\Omega,R^n)$ be a generalized solution of the Beltrami system (3.8), then there exists p_1,p_2, $p_1<n<p_2$, such that every generalized solution $f\in W^{1,p_1}_{loc}(\Omega,R^n)$ to (3.8) actually belongs to $W^{1,p_2}_{loc}(\Omega,R^n)$.*

Proof. This corollary is a direct consequence of Theorem 2.1, Theorem 3.3 and Lemma 1.3.

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