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Convergence of Nonlinear Recurrence Relations with Threshold Control and $2k-$ Periodic Coefficients

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Abstract

An nonlinear recurrence involving a piecewise constant McCulloch-Pitts function and $2k-$periodic coefficient sequences is investigated. It is found that each solution tends to $\langle -1 \rangle$ or $\langle 1 \rangle$, depending on whether the parameter λ varies from $-\infty$ to $+\infty$. We hope that our results will be useful in understanding interacting network models involving piecewise constant control functions.

Mathematics Subject Classification: xxxxx

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1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$. In [1], Zhu and Huang discussed the "limit cycle" of recurrence relation

$$x_n = ax_{n-2} + bf_\lambda(x_{n-1}), n \in \mathbb{N}, \tag{1}$$

where $a \in (0, 1), b = 1 - a$. And $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear threshold function of the form

$$f_\lambda(x) = \begin{cases} 1 & x \in (0, \lambda] \\ 0 & x \in (-\infty, 0] \bigcup (\lambda, +\infty) \end{cases}, \tag{2}$$

in which λ is a constant which acts as a threshold, through analysis get the convergence of solutions and the existence of asymptotically stable periodic solutions.

Yet in real life models, the coefficients a and b, since they are a part of the control mechanism, can rarely be kept constants. They may became time dependent and show periodic behaviors. For this reason, in [2], the authors discussed the limit cycles of the following difference equation

$$x_n = a_n x_{n-2} + b_n f_\lambda(x_{n-1}), n \in \mathbb{N}, \tag{3}$$

where $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are 2- periodic sequences with $a_i \in (0,1), b_i \in (0,+\infty), i = 0,1$. And $f : \mathbb{R} \to \mathbb{R}$ is defined by (2), and by the transform $x_{2n} = y_n, x_{2n+1} = z_n$ for $n \in \{-1, 0, \cdots\}$, the above equation can be converted into the following 2−dimensional autonomous dynamical system

$$\begin{cases} y_n = a_0 y_{n-1} + b_0 f_\lambda(z_{n-1}) \\ z_n = a_1 z_{n-1} + b_1 f_\lambda(y_n) \end{cases}, \tag{4}$$

in which the positive number λ can be regarded as a threshold bifurcation parameter. By induction, all solutions of (3) from $(-\infty, 0]^2$ tend to the point (0,0), all solutions of (3) from $\mathbb{R}^2/(-\infty, 0]^2$ tend to the point $(\frac{b_0}{1-a_0}, 0), (0, \frac{b_1}{1-a_1})$, or $(\frac{b_0}{1-a_0}, \frac{b_1}{1-a_1})$. In [3], the authors discussed the limit cycles of the following difference equation

$$x_n = a_n x_{n-2} + b_n f_\lambda(x_{n-1}), n \in \mathbb{N}, \tag{5}$$

where $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are $2k$-periodic sequences with $a_i \in (0,1)$, $b_i = 1 - a_i$, $i = 0, 1, \cdots, 2k-1$. And f satisfies

$$f_\lambda(x) = \begin{cases} 1 & x \in (0, \lambda] \\ 0 & x \in (-\infty, 0] \bigcup (\lambda, +\infty) \end{cases},$$

in which the number λ can be regarded as a threshold bifurcation parameter.

By induction, the authors deduce (bifurcation) result such as the following. If $0 < \lambda < 1$, then all solutions$\{(x_n, x_{n+1})\}_{n=-2}^{\infty}$ which originated from the positive orthant approach a limit 2-cycles; if $\lambda > 1$, then all solutions that originated from the positive orthant tend towards the limit 1-cycle(1,1); if $\lambda = 1$, then all solutions originated from the positive orthant tend towards the limit 1-cycle(1,1) or 2-cycles(1,0) or (0,1).

This paper mainly studies the following form of nonlinear difference equation

$$x_n = a_n x_{n-2} + b_n f_\lambda(x_{n-1}), n \in \mathbb{N}, \tag{6}$$

where $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are $2k$-periodic sequences with $a_i \in (0,1)$, $b_i = 1 - a_i$, $i = 0, 1, 2, \cdots, 2k-1$, f satisfies

$$f_\lambda(x) = \begin{cases} 1 & x \in (\lambda, +\infty) \\ -1 & x \in (-\infty, \lambda] \end{cases},$$

in which the number λ can be regarded as a threshold bifurcation parameter.

In order to study the asymptotic behavior of (6), let us first note that it is a three-term recurrence relation so that, given x_{-2} and x_{-1}, we may calculate x_0, x_1, x_2, and so forth in a sequential manner. The resulting sequence $x = \{x_n\}_{n=-2}^{\infty}$ is naturally called a solution of (6). For example, when $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are 4-periodic sequences, we may write

$$\begin{aligned}
x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}),\\
x_1 &= a_1 x_{-1} + b_1 f_\lambda(x_0),\\
x_2 &= a_2 x_0 + b_2 f_\lambda(x_1),\\
x_3 &= a_3 x_1 + b_3 f_\lambda(x_2),\\
x_4 &= a_4 x_2 + b_4 f_\lambda(x_3) = a_0 x_2 + b_0 f_\lambda(x_3),\\
x_5 &= a_5 x_3 + b_5 f_\lambda(x_4) = a_1 x_3 + b_1 f_\lambda(x_4),\\
&\vdots
\end{aligned}$$

This motivates us to define a vector equation. Given a sequence $x = \{x_n\}_{n=a}^{\infty}$, its Casoratian vector sequence is $\{\langle x_i\rangle\}_{i=a}^{\infty}$, where $\langle x_i\rangle = col(x_i, x_{i+1}, \cdots, x_{i+2k-1}), i = a, a+1, \cdots$. Then (6) is equivalent to the asynchronous vector equation

$$\langle x_{2kn}\rangle = A\langle x_{2k(n-1)}\rangle + Bf_\lambda(\langle x_{2k(n-1)+1}\rangle), n = 0, 1, 2, \cdots, \tag{7}$$

where

$$A = \begin{pmatrix} a_0 & 0 & & 0 \\ 0 & a_1 & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & a_{2k-1} \end{pmatrix}, B = \begin{pmatrix} b_0 & 0 & & 0 \\ 0 & b_1 & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & b_{2k-1} \end{pmatrix},$$

$$f_\lambda\langle x_i\rangle = col(f_\lambda(x_i), f_\lambda(x_{i+1}), \cdots, f_\lambda(x_{i+2k-1})).$$

Note that, given (x_{-2}, x_{-1}), we may use (7) to generate $\langle x_0\rangle, \langle x_{2k}\rangle, \langle x_{4k}\rangle, \cdots$. Which, when "line up", yields the same $x_0, x_1, x_2, \cdots$ as described above. For this reason, the sequence $\{\langle x_i\rangle\}_{i=0}^{\infty}$ will be called the solution of (7) determined by (x_{-2}, x_{-1}).

Therefore, to obtain complete asymptotic behaviors of (6), we need to derive the results for solutions of (7) determined by vectors (x_{-2}, x_{-1}) in the entire plane. In the following discussion, we will allow the bifurcation parameter λ to vary from $-\infty$ to $+\infty$. For the sake of convenience, we also need to introduce some notations:

$$\delta = a_0 a_2 \cdots a_{2k-2}, \rho = a_1 a_3 \cdots a_{2k-1},$$

while the numbers

$$D_i^{(j_0, j_1, \cdots, j_m)}, E_i^{(j_0, j_1, \cdots, j_m)},$$

$$A_i^{(j_0,j_1,\cdots,j_m)}, B_i^{(j_0,j_1,\cdots,j_m)},$$

and their properties are listed in the Appendix. These numbers are introduced in order to break the plane into different parts such that the behavior of each solution of (7) which originates from each part may be traced.

Indeed, we will consider five cases: (i) $\lambda = 1$, (ii) $\lambda > 1$, (iii) $\lambda = -1$, (iv) $\lambda < -1$, and (v) $-1 < \lambda < 1$.

2 Main Results

The case where $\lambda = 1$.

Lemma 2.1 *Suppose $\lambda = 1$. If $\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2/(\lambda, +\infty)^2$, then there exists an integer $r \in \{0, 1, \cdots, 2k-2\}, j \in \mathbb{N}$ such that $(x_{2kj+r}, x_{2kj+r+1}) \in (-\infty, \lambda]^2$.*

Proof. (i). Suppose $(x_{-2}, x_{-1}) \in (-\infty, \lambda]^2$, then we are done.
(ii). Suppose $(x_{-2}, x_{-1}) \in (-\infty, \lambda] \times (\lambda, +\infty)$. For our assumption, we have $a_i\lambda + b_i = \lambda$ for $i = 0, 1, \cdots, 2k-1$. By induction,

$$\begin{aligned}
x_0 &= a_0x_{-2} + b_0f_\lambda(x_{-1}) = a_0x_{-2} + b_0 \leq a_0\lambda + b_0 = \lambda,\\
x_1 &= a_1x_{-1} + b_1f_\lambda(x_0) = a_1x_{-1} - b_1 \in \mathbb{R},\\
x_2 &= a_2x_0 + b_2f_\lambda(x_1) \leq a_2x_0 + b_2 \leq a_2\lambda + b_2 = \lambda,\\
&\vdots\\
x_{2ki+2m} &= a_{2m}x_{2ki+2m-2} + b_{2m}f_\lambda(x_{2ki+2m-1}) \leq a_{2m}\lambda + b_{2m} = \lambda.
\end{aligned}$$

We see that $x_{2ki+2m} \in (-\infty, \lambda]$ for any $m \in \{0, 1, \cdots, k-1\}$ and $i \in \mathbb{N}$; and hence,

$$\begin{aligned}
x_{2ki+2m+1} &= a_{2m+1}x_{2ki+2m-1} + b_{2m+1}f_\lambda(x_{2ki+2m})\\
&= a_{2m+1}x_{2ki+2m-1} - b_{2m+1}\\
&= a_{2m+1}(a_{2m-1}x_{2ki+2m-3} - b_{2m-1}) - b_{2m+1}\\
&= a_{2m+1}a_{2m-1}\cdots a_1\rho^i x_{-1} + a_{2m+1}a_{2m-1}\cdots a_1\rho^i - 1.
\end{aligned}$$

Thus, $\lim_{i\to+\infty} x_{2ki+2m+1} = -1 \in (-\infty, \lambda]$ for any $m \in \{0, 1, \cdots, k-1\}$, then there exists enough large $j \in \mathbb{N}$ such that $(x_{2kj}, x_{2kj+1}) \in (-\infty, \lambda]^2$.
(iii). Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (-\infty, \lambda]$. By induction, we may see that $x_{2ki+2m-1} \in (-\infty, \lambda]$ for any $m \in \{0, 1, \cdots, k\}$ and $i \in \mathbb{N}$; and hence,

$$\begin{aligned}
x_{2ki+2m} &= a_{2m}x_{2ki+2m-2} + b_{2m}f_\lambda(x_{2ki+2m-1})\\
&= a_{2m}x_{2ki+2m-2} - b_{2m}\\
&= a_{2m}(a_{2m-2}x_{2ki+2m-4} - b_{2m-2}) - b_{2m}\\
&= a_{2m}a_{2m-2}\cdots a_0\delta^i x_{-2} + a_{2m}a_{2m-2}\cdots a_0\delta^i - 1.
\end{aligned}$$

Thus, $\lim_{i\to+\infty} x_{2ki+2m} = -1 \in (-\infty, \lambda]$ for any $m \in \{0, 1, \cdots, k-1\}$, then there exists enough large $j \in \mathbb{N}$ such that $(x_{2kj}, x_{2kj+1}) \in (-\infty, \lambda]^2$.
By (i), (ii), (iii) the proof is complete.

Theorem 2.2 *Suppose $\lambda = 1$. The solution $\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty}$ of (7) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2/(\lambda, +\infty)^2$ will tend to $\langle -1\rangle$.*

Proof. In view of Lemma 1, we may assume without loss of generality that $(x_{-2}, x_{-1}) \in (-\infty, \lambda]^2$. For our assumption, we have $a_i\lambda - b_i < \lambda$ for $i = 0, 1, \cdots, 2k-1$. Furthermore, by induction, we have,

$$\begin{aligned}
x_0 &= a_0x_{-2} + b_0f_\lambda(x_{-1}) = a_0x_{-2} - b_0 \le a_0\lambda - b_0 < \lambda,\\
x_1 &= a_1x_{-1} + b_1f_\lambda(x_0) = a_1x_{-1} - b_1 \le a_1\lambda - b_1 < \lambda,\\
x_2 &= a_2x_0 + b_2f_\lambda(x_1) = a_2x_0 - b_2 \le a_2\lambda - b_2 < \lambda,\\
&\vdots\\
x_{2k-1} &= a_{2k-1}x_{2k-3} + b_{2k-1}f_\lambda(x_{2k-2}) \le a_{2k-1}\lambda - b_{2k-1} < \lambda.
\end{aligned}$$

We see that $x_{2ki+j} = a_jx_{2ki+j-2} - b_j \in (-\infty, \lambda]$ for any $j \in \{0, 1, \cdots, 2k-1\}$ and $i \in \mathbb{N}$; and hence,

$$\begin{aligned}
\langle x_{2kn}\rangle &= A\langle x_{2k(n-1)}\rangle - B\\
&= A^n\langle x_0\rangle - A^{n-1}B - \cdots - B\\
&= A^n\langle x_0\rangle + A^n - E.
\end{aligned}$$

Since $\langle x_0\rangle \in (-\infty, \lambda]$ and A^n tends towards 0 as n tends towards $+\infty$, we see that $\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty}$ tends towards $\langle -1\rangle$. The proof is complete.

Theorem 2.3 *Suppose $\lambda = 1$. The solution $\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty}$ of (7) with $(x_{-2}, x_{-1}) \in (\lambda, +\infty)^2$ will tend to $\langle 1\rangle$.*

Proof. For our assumption, we have $a_i\lambda + b_i = \lambda$ for $i = 0, 1, \cdots, 2k-1$. Furthermore, by induction, we have,

$$\begin{aligned}
x_0 &= a_0x_{-2} + b_0f_\lambda(x_{-1}) = a_0x_{-2} + b_0 > a_0\lambda + b_0 = \lambda,\\
x_1 &= a_1x_{-1} + b_1f_\lambda(x_0) = a_1x_{-1} + b_1 > a_1\lambda + b_1 = \lambda,\\
x_2 &= a_2x_0 + b_2f_\lambda(x_1) = a_2x_0 + b_2 > a_2\lambda + b_2 = \lambda,\\
&\vdots\\
x_{2k-1} &= a_{2k-1}x_{2k-3} + b_{2k-1}f_\lambda(x_{2k-2}) > a_{2k-1}\lambda + b_{2k-1} = \lambda.
\end{aligned}$$

We see that $x_{2ki+j} = a_jx_{2ki+j-2} + b_j \in (\lambda, +\infty)$ for any $j \in \{0, 1, \cdots, 2k-1\}$ and $i \in \mathbb{N}$; and hence,

$$\begin{aligned}
\langle x_{2kn}\rangle &= A\langle x_{2k(n-1)}\rangle + B\\
&= A^n\langle x_0\rangle - A^{n-1}B + \cdots + B\\
&= A^n\langle x_0\rangle - A^n + E.
\end{aligned}$$

Since $\langle x_0 \rangle \in (\lambda, +\infty)$ and A^n tends towards 0 as n tends towards $+\infty$, we see that $\{(\langle x_{2kn} \rangle)\}_{n=0}^{\infty}$ tends towards $\langle 1 \rangle$. The proof is complete.

The case where $\lambda > 1$.

Lemma 2.4 *Suppose $\lambda > 1$. If $\{(\langle x_{2kn} \rangle)\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2$, then there exists an integer $r \in \{0, 1, \cdots, 2k-2\}, j \in \mathbb{N}$ such that $(x_{2kj+r}, x_{2kj+r+1}) \in (-\infty, \lambda]^2$.*

Proof. (i). Suppose $(x_{-2}, x_{-1}) \in (-\infty, \lambda]^2$, then we are done.

(ii). Suppose $(x_{-2}, x_{-1}) \in (-\infty, \lambda] \times (\lambda, +\infty) \bigcup (\lambda, +\infty) \times (-\infty, \lambda]$. By induction, the proof are similar to (ii), (iii) of Lemma 1, and hence omitted.

(iii). Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty)^2$, if $x_{2ki+m} \in (\lambda, +\infty)$ for all $m \in \{0, 1, \cdots, 2k-1\}, i \in \mathbb{N}$, then $x_{2ki+m} = a_m x_{2ki+m-2} + b_m$ for all $m \in \{0, 1, \cdots, 2k-2\}$, thus $\lim_{i \to +\infty} x_{2ki+m} = 1 \in (-\infty, \lambda]$, which is a contradiction. Therefore, there exist $j \in \mathbb{N}$, such that $x_{2kj} \in (-\infty, \lambda], x_0, x_1, \cdots, x_{2k-1}, x_{2k}, x_{2k+1}, \cdots, x_{4k-1}, \cdots, x_{2kj-2}, x_{2kj-1} \in (\lambda, +\infty)$. So by (i), (ii), (iii) conclusion holds. The proof is complete.

Theorem 2.5 *Suppose $\lambda > 1$. The solution $\{(\langle x_{2kn} \rangle)\}_{n=0}^{\infty}$ of (7) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2$ will tend to $\langle -1 \rangle$.*

In view of Lemma 2, we may assume without loss of generality that $(x_{-2}, x_{-1}) \in (-\infty, \lambda]^2$. For our assumption, we have $a_i \lambda - b_i < \lambda$ for $i = 0, 1, \cdots, 2k-1$. So the proof is same as Theorem 1 and is skipped.

The case where $\lambda = -1$.

By arguments similar to those in the lemma 2, we may show the following result. The case $\lambda = -1$ is similar to $\lambda = 1$, the proof of Lemma 3, Theorem 4 and Theorem 5, we can refer to Lemma 1, Theorem 1 and Theorem 2.

Lemma 2.6 *Suppose $\lambda = -1$. If $\{(\langle x_{2kn} \rangle)\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2/(-\infty, \lambda]^2$, then there exists an integer $r \in \{0, 1, \cdots, 2k-2\}, j \in \mathbb{N}$ such that $(x_{2kj+r}, x_{2kj+r+1}) \in (\lambda, +\infty)^2$.*

Proof. For our assumption, we have $a_i \lambda + b_i > a_i \lambda - b_i = \lambda$ for $i = 0, 1, \cdots, 2k-1$.

(i). Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty)^2$, then we are done.

(ii). Suppose $(x_{-2}, x_{-1}) \in (-\infty, \lambda] \times (\lambda, +\infty)$. By induction, we see that $x_{2ki+2m-1} \in (\lambda, +\infty)$ for any $m \in \{0, 1, \cdots, k\}$ and $i \in \mathbb{N}$; and hence,

$$\begin{aligned} x_{2ki+2m} &= a_{2m} x_{2ki+2m-2} + b_{2m} f_\lambda(x_{2ki+2m-1}) \\ &= a_{2m} x_{2ki+2m-2} + b_{2m} \\ &= a_{2m}(a_{2m-2} x_{2ki+2m-4} + b_{2m-2}) + b_{2m} \\ &= a_{2m} a_{2m-2} \cdots a_0 \delta^i x_{-2} - a_{2m} a_{2m-2} \cdots a_0 \delta^i + 1. \end{aligned}$$

Thus, $\lim_{i\to+\infty} x_{2ki+2m+1} = 1 \in (\lambda, +\infty)$ for any $m \in \{0, 1, \cdots, k-1\}$, then there exists enough large $j \in \mathbb{N}$ such that $(x_{2kj}, x_{2kj+1}) \in (\lambda, +\infty)^2$.

(iii). Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (-\infty, \lambda]$. By induction, we may see that $x_{2ki+2m} \in (\lambda, +\infty)$ for any $m \in \{0, 1, \cdots, k-1\}$ and $i \in \mathbb{N}$; and hence,

$$\begin{aligned} x_{2ki+2m+1} &= a_{2m+1}x_{2ki+2m-1} + b_{2m+1}f_\lambda(x_{2ki+2m}) \\ &= a_{2m+1}x_{2ki+2m-1} + b_{2m+1} \\ &= a_{2m+1}(a_{2m-1}x_{2ki+2m-3} + b_{2m-1}) + b_{2m+1} \\ &= a_{2m+1}a_{2m-1}\cdots a_1\rho^i x_{-1} - a_{2m+1}a_{2m-1}\cdots a_1\rho^i + 1. \end{aligned}$$

Thus, $\lim_{i\to+\infty} x_{2ki+2m+1} = 1 \in (\lambda, +\infty)$ for any $m \in \{0, 1, \cdots, k-1\}$, then there exists enough large $j \in \mathbb{N}$ such that $(x_{2kj}, x_{2kj+1}) \in (\lambda, +\infty)^2$.
By (i), (ii), (iii) the proof is complete.

Theorem 2.7 *Suppose $\lambda = -1$. The solution $\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty}$ of (7) with $(x_{-2}, x_{-1}) \in (-\infty, \lambda]^2$ will tend to $\langle -1\rangle$.*

Proof. For our assumption, we have $a_i\lambda - b_i = \lambda$ for $i = 0, 1, \cdots, 2k-1$. Furthermore, by induction, we see that $x_{2ki+j} = a_j x_{2ki+j-2} - b_j \in (-\infty, \lambda]$ for any $j \in \{0, 1, \cdots, 2k-1\}$ and $i \in \mathbb{N}$; and hence, the proof is similar as Theorem 1 and is skipped.

Theorem 2.8 *Suppose $\lambda = -1$. The solution $\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty}$ of (7) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2/(-\infty, \lambda]^2$ will tend to $\langle 1\rangle$.*

Proof. In view of Lemma 3, we may assume without loss of generality that $(x_{-2}, x_{-1}) \in (\lambda, +\infty)^2$. For our assumption, we have $a_i\lambda + b_i > \lambda$ for $i = 0, 1, \cdots, 2k-1$. Furthermore, by induction, the proof is similar as Theorem 2 and is skipped.

The case where $\lambda < -1$.

Lemma 2.9 *Suppose $\lambda < -1$. If $\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2$, then there exists an integer $r \in \{0, 1, \cdots, 2k-2\}, j \in \mathbb{N}$ such that $(x_{2kj+r}, x_{2kj+r+1}) \in (\lambda, +\infty)^2$.*

Proof. (i). Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty)^2$, then we are done.

(ii). Suppose $(x_{-2}, x_{-1}) \in (-\infty, \lambda] \times (\lambda, +\infty) \bigcup (\lambda, +\infty) \times (-\infty, \lambda]$. By induction, the proof are similar to (ii), (iii) of Lemma 3, and hence omitted.

(iii). Suppose $(x_{-2}, x_{-1}) \in (-\infty, \lambda]^2$, if $x_{2ki+m} \in (-\infty, \lambda]$ for all $m \in \{0, 1, \cdots, 2k-1\}, i \in \mathbb{N}$, then $x_{2ki+m} = a_m x_{2ki+m-2} - b_m$ for all $m \in \{0, 1, \cdots, 2k-1\}$, thus $\lim_{i\to+\infty} x_{2ki+m} = -1 \in (\lambda, +\infty)$, which is a contradiction. Therefore, there exist $j \in \mathbb{N}$, such that $x_{2kj} \in (\lambda, +\infty), x_0, x_1, \cdots,$
$x_{2k-1}, x_{2k}, x_{2k+1}, \cdots, x_{4k-1}, \cdots, x_{2kj-2}, x_{2kj-1} \in (-\infty, \lambda]$. So by (i), (ii), (iii) conclusion holds. The proof is complete.

Theorem 2.10 *Suppose* $\lambda < -1$. *The solution* $\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty}$ *of (7) with* $(x_{-2}, x_{-1}) \in \mathbb{R}^2$ *will tend to* $\langle 1\rangle$.

For our assumption, we have $a_i\lambda + b_i > \lambda$ for $i = 0, 1, \cdots, 2k-1$. In view of Lemma 4, we may assume without loss of generality that $(x_{-2}, x_{-1}) \in (\lambda, +\infty)^2$. So the proof is similar as Theorem 2 and is skipped.

The case where $-1 < \lambda < 1$.

By arguments similar to those in the Theorem 5 and Theorem 6, we may show the following two results.

Theorem 2.11 *Suppose* $-1 < \lambda < 1$, *then the following conclusions hold.*
(i). Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty)^2$, *the every solution of (7) tend to* $\langle 1\rangle$.
(ii). Suppose $(x_{-2}, x_{-1}) \in (-\infty, \lambda]^2$, *the every solution of (7) tend to* $\langle -1\rangle$.

For our assumption, we have $a_i\lambda + b_i > \lambda > a_i\lambda - b_i$ for $i = 0, 1, \cdots, 2k-1$. So (i) and (ii) respectively are similar Theorem 2 and Theorem 1. Hence the proofs are omitted.

Theorem 2.12 *Suppose* $-1 < \lambda < 1$. *Suppose that* $\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty}$ *is a solution of (7) with* $(x_{-2}, x_{-1}) \in (D_i^{(0,2,\cdots,2m-2)}, D_i^{(0,2,\cdots,2m)}] \times (L_j^{(1,3,\cdots,2l+1)}, L_j^{(1,3,\cdots,2l-1)}]$, *where* $i, j \in \{0, 1, \cdots\}, m, l \in \{0, 1, \cdots, k-1\}$, *then,*
(i).$\lim_{i\to\infty}\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty} = \langle -1\rangle$ *for* $i = j, 0 \le m \le l$.
(ii).$\lim_{i\to\infty}\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty} = \langle 1\rangle$ *for* $i = j, m > l$.
(iii).$\lim_{i\to\infty}\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty} = \langle -1\rangle$ *for* $i < j$.
(iv).$\lim_{i\to\infty}\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty} = \langle 1\rangle$ *for* $i > j$.

Proof. Suppose that $(x_{-2}, x_{-1}) \in (D_i^{(0,2,\cdots,2m-2)}, D_i^{(0,2,\cdots,2m)}] \times (L_j^{(1,3,\cdots,2l+1)}, L_j^{(1,3,\cdots,2l-1)}]$, then,

(i)We distinguish two different cases.

Case 1. Consider $0 = i = j, 0 \le m \le l$. Then, by induction, we have,

$$\begin{aligned}
x_0 &= a_0x_{-2} + b_0f_\lambda(x_{-1}) = a_0x_{-2} - b_0 \in (D_0^{(2,4,\cdots,2m-2)}, D_0^{(2,4,\cdots,2m)}],\\
x_1 &= a_1x_{-1} + b_1f_\lambda(x_0) = a_1x_{-1} + b_1 \in (L_0^{(3,5,\cdots,2l+1)}, L_0^{(3,5,\cdots,2l-1)}],\\
&\vdots\\
x_{2m} &= a_{2m}x_{2m-2} + b_{2m}f_\lambda(x_{2m-1}) = a_{2m}x_{2m-2} - b_{2m} \in (a_{2m}D_0^{(-2)} - b_{2m}, D_0^{(-2)}],\\
x_{2m+1} &= a_{2m+1}x_{2m-1} + b_{2m+1}f_\lambda(x_{2m-1}) \le a_{2m+1}\lambda - b_{2m+1} < \lambda.
\end{aligned}$$

Case 2. Consider $0 < i = j, 0 \leq m \leq l$. Then, by induction, we have,

$$x_0 = a_0 x_{-2} + b_0 f_\lambda(x_{-1}) = a_0 x_{-2} - b_0 \in (D_i^{(2,4,\cdots,2m-2)}, D_i^{(2,4,\cdots,2m)}],$$
$$x_1 = a_1 x_{-1} + b_1 f_\lambda(x_0) = a_1 x_{-1} + b_1 \in (L_i^{(3,5,\cdots,2l+1)}, L_i^{(3,5,\cdots,2l-1)}],$$
$$\vdots$$
$$x_{2k-2} = a_{2k-2} x_{2k-4} + b_{2k-2} f_\lambda(x_{2k-3}) = a_{2k-2} x_{2k-4} - b_{2k-2} \in (D_{i-1}^{(0,2,\cdots,2m-2)}, D_{i-1}^{(0,2,\cdots,2m)}],$$
$$x_{2k-1} = a_{2k-1} x_{2k-3} + b_{2k-1} f_\lambda(x_{2k-2}) = a_{2k-1} x_{2k-3} + b_{2k-1} \in (L_{j-1}^{(1,3,\cdots,2l+1)}, L_{j-1}^{(1,3,\cdots,2l-1)}].$$

and by induction, we have,

$$x_{2ki} = a_0 x_{2ki-2} + b_0 f_\lambda(x_{2ki-1}) = a_0 x_{2ki-2} - b_0 \in (D_0^{(2,4,\cdots,2m-2)}, D_0^{(2,4,\cdots,2m)}],$$
$$x_{2ki+1} = a_1 x_{2ki-1} + b_1 f_\lambda(x_{2ki}) = a_1 x_{2ki-1} + b_1 \in (L_0^{(3,5,\cdots,2l+1)}, L_0^{(3,5,\cdots,2l-1)}],$$
$$\vdots$$
$$x_{2ki+2m} = a_{2m} x_{2ki+2m-2} - b_{2m} \in (a_{2m} D_0^{(-2)} - b_{2m}, D_0^{(-2)}],$$
$$x_{2ki+2m+1} = a_{2m+1} x_{2ki+2m-1} + b_{2m+1} f_\lambda(x_{2ki+2m}) \leq a_{2m+1}\lambda - b_{2m+1} < \lambda.$$

So by Theorem 7, we can get $\lim_{i\to\infty}\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty} = \langle -1\rangle$.

(ii) similar to (i), by distinguishing two different cases and by induction, we have the following:

Case 1. Consider $0 = i = j, m > l$.

Case 2. Consider $0 < i = j, m > l$.

The proof in detail please see (i).

(iii) We distinguish four different cases.

Case 1. Consider $0 = i < j, 0 \leq m \leq l$. Then, by induction, we have,

$$x_0 = a_0 x_{-2} + b_0 f_\lambda(x_{-1}) = a_0 x_{-2} - b_0 \in (D_0^{(2,4,\cdots,2m-2)}, D_0^{(2,4,\cdots,2m)}],$$
$$x_1 = a_1 x_{-1} + b_1 f_\lambda(x_0) = a_1 x_{-1} + b_1 \in (L_j^{(3,5,\cdots,2l+1)}, L_j^{(3,5,\cdots,2l-1)}],$$
$$\vdots$$
$$x_{2m} = a_{2m} x_{2m-2} + b_{2m} f_\lambda(x_{2m-1}) = a_{2m} x_{2m-2} - b_{2m} \in (a_{2m} D_0^{(-2)} - b_{2m}, D_0^{(-2)}],$$
$$x_{2m+1} = a_{2m+1} x_{2m-1} + b_{2m+1} f_\lambda(x_{2m}) = a_{2m+1} x_{2m-1} - b_{2m+1} \leq a_{2m+1}\lambda - b_{2m+1} < \lambda.$$

Case 2. Consider $0 = i < j, m > l$. Then, by induction, we have,

$$x_0 = a_0 x_{-2} + b_0 f_\lambda(x_{-1}) = a_0 x_{-2} - b_0 \in (D_0^{(2,4,\cdots,2m-2)}, D_0^{(2,4,\cdots,2m)}],$$
$$x_1 = a_1 x_{-1} + b_1 f_\lambda(x_0) = a_1 x_{-1} + b_1 \in (L_j^{(3,5,\cdots,2l+1)}, L_j^{(3,5,\cdots,2l-1)}],$$
$$\vdots$$
$$x_{2m} = a_{2m} x_{2m-2} + b_{2m} f_\lambda(x_{2m-1}) = a_{2m} x_{2m-2} - b_{2m} \in (a_{2m} D_0^{(-2)} - b_{2m}, D_0^{(-2)}],$$
$$x_{2m+1} = a_{2m+1} x_{2m-1} + b_{2m+1} f_\lambda(x_{2m}) = a_{2m+1} x_{2m-1} - b_{2m+1} \leq a_{2m+1}\lambda - b_{2m+1} < \lambda.$$

Case 3. Consider $0 < i < j, 0 \leq m \leq l$. Then, by induction, we have,

$$\begin{aligned}
x_0 &= a_0x_{-2} + b_0f_\lambda(x_{-1}) = a_0x_{-2} - b_0 \in (D_i^{(2,4,\cdots,2m-2)}, D_i^{(2,4,\cdots,2m)}],\\
x_1 &= a_1x_{-1} + b_1f_\lambda(x_0) = a_1x_{-1} + b_1 \in (L_i^{(3,5,\cdots,2l+1)}, L_i^{(3,5,\cdots,2l-1)}],\\
&\vdots\\
x_{2k-2} &= a_{2k-2}x_{2k-4} + b_{2k-2}f_\lambda(x_{2k-3}) = a_{2k-2}x_{2k-4} - b_{2k-2} \in (D_{i-1}^{(0,2,\cdots,2m-2)}, D_{i-1}^{(0,2,\cdots,2m)}],\\
x_{2k-1} &= a_{2k-1}x_{2k-3} + b_{2k-1}f_\lambda(x_{2k-2}) = a_{2k-1}x_{2k-3} + b_{2k-1} \in (L_{j-1}^{(1,3,\cdots,2l+1)}, L_{j-1}^{(1,3,\cdots,2l-1)}].
\end{aligned}$$

and by induction, we have,

$$\begin{aligned}
x_{2ki} &= a_0x_{2ki-2} + b_0f_\lambda(x_{2ki-1}) = a_0x_{2ki-2} - b_0 \in (D_0^{(2,4,\cdots,2m-2)}, D_0^{(2,4,\cdots,2m)}],\\
x_{2ki+1} &= a_1x_{2ki-1} + b_1f_\lambda(x_{2ki}) = a_1x_{2ki-1} + b_1 \in (L_{j-i}^{(3,5,\cdots,2l+1)}, L_{j-i}^{(3,5,\cdots,2l-1)}],\\
&\vdots\\
x_{2ki+2m} &= a_{2m}x_{2ki+2m-2} - b_{2m} \in (a_{2m}D_0^{(-2)} - b_{2m}, D_0^{(-2)}],\\
x_{2ki+2m+1} &= a_{2m+1}x_{2ki+2m-1} + b_{2m+1}f_\lambda(x_{2ki+2m}) \leq a_{2m+1}\lambda - b_{2m+1} < \lambda.
\end{aligned}$$

Case 4. Consider $0 < i < j, m > l$. Then, by induction, we have,

$$\begin{aligned}
x_0 &= a_0x_{-2} + b_0f_\lambda(x_{-1}) = a_0x_{-2} - b_0 \in (D_i^{(2,4,\cdots,2m-2)}, D_i^{(2,4,\cdots,2m)}],\\
x_1 &= a_1x_{-1} + b_1f_\lambda(x_0) = a_1x_{-1} + b_1 \in (L_i^{(3,5,\cdots,2l+1)}, L_i^{(3,5,\cdots,2l-1)}],\\
&\vdots\\
x_{2k-2} &= a_{2k-2}x_{2k-4} + b_{2k-2}f_\lambda(x_{2k-3}) = a_{2k-2}x_{2k-4} - b_{2k-2} \in (D_{i-1}^{(0,2,\cdots,2m-2)}, D_{i-1}^{(0,2,\cdots,2m)}],\\
x_{2k-1} &= a_{2k-1}x_{2k-3} + b_{2k-1}f_\lambda(x_{2k-2}) = a_{2k-1}x_{2k-3} + b_{2k-1} \in (L_{j-1}^{(1,3,\cdots,2l+1)}, L_{j-1}^{(1,3,\cdots,2l-1)}].
\end{aligned}$$

and by induction, we have,

$$\begin{aligned}
x_{2ki} &= a_0x_{2ki-2} + b_0f_\lambda(x_{2ki-1}) = a_0x_{2ki-2} - b_0 \in (D_0^{(2,4,\cdots,2m-2)}, D_0^{(2,4,\cdots,2m)}],\\
x_{2ki+1} &= a_1x_{2ki-1} + b_1f_\lambda(x_{2ki}) = a_1x_{2ki-1} + b_1 \in (L_{j-i}^{(3,5,\cdots,2l+1)}, L_{j-i}^{(3,5,\cdots,2l-1)}],\\
&\vdots\\
x_{2ki+2m-1} &= a_{2m-1}x_{2ki+2m-3} + b_{2m-1}f_\lambda(x_{2ki+2m-2}) = a_{2m-1}x_{2ki+2m-3} - b_{2m-1}\\
&\in (L_{j-i}^{(1,3,5,\cdots,2l+1,2m+1,2m+3,\cdots,2k-1)}, L_{j-i}^{(1,3,5,\cdots,2l-1,2m+1,2m+3,\cdots,2k-1)}],\\
x_{2ki+2m} &= a_{2m}x_{2ki+2m-2} - b_{2m} \in (a_{2m}D_0^{(-2)} - b_{2m}, D_0^{(-2)}],\\
x_{2ki+2m+1} &= a_{2m+1}x_{2ki+2m-1} + b_{2m+1}f_\lambda(x_{2ki+2m}) \leq a_{2m+1}\lambda - b_{2m+1} < \lambda.
\end{aligned}$$

So by Theorem 7, we can get $\lim_{i\to\infty}\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty} = \langle -1\rangle$.

(iv) similar to (iii), by distinguishing four different cases and by induction, we have the following:

Case 1. Consider $0 = j < i, 0 \leq m \leq l$.

Case 2. Consider $0 = j < i, m > l$.

Case 3. Consider $0 < j < i, 0 \leq m \leq l$.

Case 4. Consider $0 = j < i, m > l$.

The proof in detail please see (iii).

Theorem 2.13 *Suppose $-1 < \lambda < 1$. Suppose that $\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in (A_i^{(0,2,\cdots,2m)}, A_i^{(0,2,\cdots,2m-2)}] \times (B_j^{(1,3,\cdots,2l-1)}, B_j^{(1,3,\cdots,2l+1)}]$, where $i, j \in \{0, 1, \cdots\}, m, l \in \{0, 1, \cdots, k-1\}$, then,*

(i).$\lim_{i\to\infty}\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty} = \langle 1\rangle$ for $i = j, 0 \leq m \leq l$.

(ii).$\lim_{i\to\infty}\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty} = \langle -1\rangle$ for $i = j, m > l$.

(iii).$\lim_{i\to\infty}\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty} = \langle 1\rangle$ for $i < j$.

(iv).$\lim_{i\to\infty}\{(\langle x_{2kn}\rangle)\}_{n=0}^{\infty} = \langle -1\rangle$ for $i > j$.

The proof is similar Theorem 8. Hence the proofs are omitted.

3 Discussion

The result in the previous section for the system (7) can easily be translated into result for (6). We summa as follow:

(1) Suppose $\lambda = 1$. A solution $\{(x_n, x_{n+1})\}_{n=-2}^{\infty}$ of (6) with $(x_{-2}, x_{-1}) \in (\lambda, +\infty)^2$ will tend towards $\langle 1\rangle$. If $(x_{-2}, x_{-1}) \in \mathbb{R}^2/(\lambda, +\infty)^2$, the solutions will tend towards $\langle -1\rangle$.

(2) Suppose $\lambda > 1$. A solution $\{(x_n, x_{n+1})\}_{n=-2}^{\infty}$ of (6) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2$ will eventually fall into $(-\infty, \lambda]^2$ and approach $\langle -1\rangle$.

(3) Suppose $\lambda = -1$. A solution $\{(x_n, x_{n+1})\}_{n=-2}^{\infty}$ of (6) with $(x_{-2}, x_{-1}) \in (-\infty, \lambda]^2$ will tend towards $\langle -1\rangle$. If $(x_{-2}, x_{-1}) \in \mathbb{R}^2/(-\infty, \lambda]^2$, the solutions will tend towards $\langle 1\rangle$.

(4) Suppose $\lambda < -1$. A solution $\{(x_n, x_{n+1})\}_{n=-2}^{\infty}$ of (6) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2$ will eventually fall into $(\lambda, +\infty)^2$ and approach $\langle 1\rangle$.

(5) Suppose $-1 < \lambda < 1$. A solution $\{(x_n, x_{n+1})\}_{n=-2}^{\infty}$ of (6) with $(x_{-2}, x_{-1}) \in (\lambda, +\infty]^2$ will tend towards $\langle 1\rangle$. If $(x_{-2}, x_{-1}) \in (-\infty, \lambda]^2$, the solutions will tend towards $\langle -1\rangle$. If $(x_{-2}, x_{-1}) \in \mathbb{R}^2/(-\infty, \lambda]^2/(\lambda, +\infty)^2$, results in detail please see Theorem 8 and Theorem 9.

In neural network terminologies, we have discussed a simple neuron recurrent McCulloch-Pitts-type neural network with a threshold and $2k$-periodic coefficients. Such an observation seems to appear in many natural processes and hence our model may be use to explain such phenomena. It is also expected that when a group of neural units interact with each other in a network where each unit is governed by evolutionary laws of the form (6), complex but manageable analytical results can be obtained. These will be left to other studiers in the future.

4 Appendix

We let

$$\langle 1\rangle = (1, 1, \cdots, 1), \langle -1\rangle = (-1, -1, \cdots, -1).$$

$$D_i^{(j_0,j_1,\cdots,j_m)} = \frac{\lambda + 1 - a_{j_0}a_{j_1}\cdots a_{j_m}\delta^i}{a_{j_0}a_{j_1}\cdots a_{j_m}\delta^i}, j_0, j_1, \cdots, j_m \in \{0, 2, \cdots, 2k-2\} \quad (8)$$

$$B_i^{(j_0,j_1,\cdots,j_m)} = \frac{\lambda + 1 - a_{j_0}a_{j_1}\cdots a_{j_m}\rho^i}{a_{j_0}a_{j_1}\cdots a_{j_m}\rho^i}, j_0, j_1, \cdots, j_m \in \{1, 3, \cdots, 2k-1\} \quad (9)$$

$$A_i^{(j_0,j_1,\cdots,j_m)} = \frac{\lambda - 1 + a_{j_0}a_{j_1}\cdots a_{j_m}\delta^i}{a_{j_0}a_{j_1}\cdots a_{j_m}\delta^i}, j_0, j_1, \cdots, j_m \in \{0, 2, \cdots, 2k-2\} \quad (10)$$

$$L_i^{(j_0,j_1,\cdots,j_m)} = \frac{\lambda - 1 + a_{j_0}a_{j_1}\cdots a_{j_m}\rho^i}{a_{j_0}a_{j_1}\cdots a_{j_m}\rho^i}, j_0, j_1, \cdots, j_m \in \{1, 3, \cdots, 2k-1\} \quad (11)$$

We assume $-1 < \lambda < 1$. Then,

$$D_0^{(-2)} = B_0^{(-1)} = \lambda.$$

$$D_i^{(0,2,\cdots,2m-2)} = D_{i+1}^{(-2)},$$

$$B_i^{(1,3,\cdots,2m-1)} = B_{i+1}^{(-1)},$$

$$D_i^{(j_0,j_1,\cdots,j_m)} < D_i^{(j_0,j_1,\cdots,j_{m+1})},$$

$$B_i^{(j_0,j_1,\cdots,j_m)} < B_i^{(j_0,j_1,\cdots,j_{m+1})},$$

$$\lim_{i\to\infty} D_i^{(j_0,j_1,\cdots,j_m)} = \lim_{i\to\infty} B_i^{(j_0,j_1,\cdots,j_m)} = +\infty.$$

And,

$$A_0^{(-2)} = L_0^{(-1)} = \lambda.$$

$$A_i^{(0,2,\cdots,2m-2)} = A_{i+1}^{(-2)},$$

$$L_i^{(1,3,\cdots,2m-1)} = L_{i+1}^{(-1)},$$

$$A_i^{(j_0,j_1,\cdots,j_{m+1})} < A_i^{(j_0,j_1,\cdots,j_m)},$$

$$L_i^{(j_0,j_1,\cdots,j_{m+1})} < L_i^{(j_0,j_1,\cdots,j_m)},$$

$$\lim_{i\to\infty} A_i^{(j_0,j_1,\cdots,j_m)} = \lim_{i\to\infty} L_i^{(j_0,j_1,\cdots,j_m)} = -\infty.$$

Thus,

$$(\lambda, +\infty) = \bigcup_{\infty}^{i=0} \bigcup_{k-1}^{m=0} (D_i^{(0,2,\cdots,2m-2)}, D_i^{(0,2,\cdots,2m)}] \quad (12)$$

$$(\lambda, +\infty) = \bigcup_{\infty}^{i=0} \bigcup_{k-1}^{m=0} (B_i^{(1,3,\cdots,2m-1)}, B_i^{(1,3,\cdots,2m+1)}] \tag{13}$$

$$(-\infty, \lambda] = \bigcup_{\infty}^{i=0} \bigcup_{k-1}^{m=0} (A_i^{(0,2,\cdots,2m)}, A_i^{(0,2,\cdots,2m-2)}] \tag{14}$$

$$(-\infty, \lambda]) = \bigcup_{\infty}^{i=0} \bigcup_{k-1}^{m=0} (L_i^{(1,3,\cdots,2m+1)}, L_i^{(1,3,\cdots,2m-1)}] \tag{15}$$

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