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A new proof of Y-J inequality

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Abstract

In this paper, we shall show a new proof of Y-J operator inequality.

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1 Introduction

A capital letter, such as T, stands for a bounded linear operator on a Hilbert space.

In [2], J. Yuan and G. Ji proved the following operator inequality for three positive operators.

Theorem 1.1(Y-J inequality, [2]). If $A \geq B \geq C \geq 0$, $0 \leq q \leq 1, r, t \geq 0$, then

$$A^{q+r} \geq [A^{\frac{r}{2}}(B^{\frac{t}{2}}C^{p}B^{\frac{t}{2}})^{s}A^{\frac{r}{2}}]^{\frac{q+r}{(p+t)s+r}} \tag{1.1}$$

holds for $p \geq 1, s \geq \frac{q}{p+t}$.

Y-J inequality plays an important role on the development of the theory of positive linear map, see [2] for details. The authors proved it by operator monotonic function. In this paper, we shall show a new proof of Y-J inequality.

In order to show our new proof, we list a famous operator inequality first.

Theorem 1.2(Furuta inequality, [1]). If $A \geq B \geq 0$, $r \geq 0$, then

$$A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} \tag{1.2}$$

holds for $p \geq 1$.

2 Main Result

In this section, we shall prove Y-J inequality by a new way.

Proof. Applying $B \geq C \geq 0$ to Furuta inequality, then

$$B^{1+t} \geq (B^{\frac{t}{2}} C^p B^{\frac{t}{2}})^{\frac{1+t}{p+t}} \tag{2.1}$$

holds due to the facts that $p \geq 1$ and $t \geq 0$.

Because $0 \leq \frac{1}{1+t} \leq 1$, applying Löwner-Heinz inequality to (2.1), we have

$$B \geq (B^{\frac{t}{2}} C^p B^{\frac{t}{2}})^{\frac{1}{p+t}}. \tag{2.2}$$

Let $A_1 = A, B_1 = (B^{\frac{t}{2}} C^p B^{\frac{t}{2}})^{\frac{1}{p+t}}$. Notice that $A_1^q = A^q \geq B^q \geq B_1^q \geq 0$, $\frac{(p+t)s}{q} \geq 1$ and $\frac{r}{q} \geq 0$. By Furuta inequality, we have

$$(A_1^q)^{1+r/q} \geq (A_1^{q\frac{r/q}{2}} B_1^{q(p+t)s/q} A_1^{q\frac{r/q}{2}})^{\frac{1+r/q}{(p+t)s/q+r/q}}. \tag{2.3}$$

Replacing A_1 by A and B_1 by $(B^{\frac{t}{2}} C^p B^{\frac{t}{2}})^{\frac{1}{p+t}}$ above, then we can obtain (1.1).

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