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Unique common fixed points in b_2 -metric spaces

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Abstract

In this paper we show the existence of common fixed points of self-mappings defined on the b_2-metric spaces. This is done by using the contractive condition and quasi-contractive condition defined via a comparison function.

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1 Introduction

Over the last fifty years, the fixed point theory has been proved to be a very powerful and important tool for the study on the nonlinear phenomena.

After the contractive principle was proved by Bnanch[1] in 1922, there appeared many other works on the fixed theory under different contractive conditions on spaces such as: quasi-metric spaces[2, 3], G-metric spaces[4], Menger spaces[5], metric-type spaces[6] and fuzzy metric spaces[7, 8, 9]. It has becomed one of the research activity centers to study the fixed points of the mappings which satisfy certain contractive or quasi-contractive condition. The follows are some concise statements about it.

The notion of a b-metric space was first introduced by Czerwik in [10, 11] and then many fixed point results were obtained for single or multi-valued mappings by Czerwik and many other authors. On the other hand, the notion of 2-metric space was introduced by Gähler in[12], having the area of a triangle in $\mathbb{R}^2$ as an inspirative example. Similarly, several fixed point results were also obtained for mappings defined on these kind of spaces[13, 14]. Later, Zead Mustafa[15] introduced a new type of generalized metric spaces, called b_2-metric spaces, as a generalization of both 2-metric and b-metric spaces. Some fixed point theorems were then raised under various contractive conditions in partially ordered b_2-metric spaces. Among these conditions there are conditions using comparison functions and almost generalized weakly contractive conditions.

The purpose of this paper is to consider the common fixed points of a family of self-mappings on the b_2-metric spaces. The method is to use the contractive or quasi-contractive condition defined by means of a comparison function.

2 Preliminary Notes

Before stating our main results, we introduce some necessary definitions as follows.

Definition 2.1. *[10, 11] Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if for all $x, y, z \in X$, the following conditions hold:*
(1). $d(x, y) = 0$ if and only if $x = y$.
(2). $d(x, y) = d(y, x)$.
(3). $d(x, z) \leq s[d(x, y) + d(y, z)]$.
In this case, the pair (X, d) is called a b-metric space.

Definition 2.2. *[12] Let X be a non-empty set and let $d : X \times X \times X \to \mathbb{R}$ be a map satisfying the following conditions:*
(1). For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.
(2). If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$.
(3). The symmetry: $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ for all $x, y, z \in X$.
(4). The rectangle inequality: $d(x, y, z) \leq d(x, y, a) + d(y, z, a) + d(z, x, a)$ for all $x, y, z, a \in X$.
Then d is called a 2-metric on X and (X, d) is called a 2-metric space.

Definition 2.3. *[15] Let X be a non-empty set, $s \geq 1$ be a real number and let d: $X \times X \times X \to \mathbb{R}$ be a map satisfying the following conditions:*
(1). For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.
(2). If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$.
(3). The symmetry: $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ for all $x, y, z \in X$.
(4). The rectangle inequality: $d(x, y, z) \leq s[d(x, y, a) + d(y, z, a) + d(z, x, a)]$

for all $x, y, z, a \in X$.
Then d is called a b_2-metric on X and (X, d) is called a b_2-metric space with parameter s. Obviously, for $s = 1$, b_2-metric reduces to 2-metric.

Definition 2.4. *[15] Let $\{x_n\}$ be a sequence in a b_2-metric space (X, d).*
(1). A sequence $\{x_n\}$ is said to be b_2-convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if for all $a \in X$, $\lim_{n\to\infty} d(x_n, x, a) = 0$.
(2). $\{x_n\}$ is Cauchy sequence if and only if $d(x_n, x_m, a) \to 0$, when $n, m \to \infty$.
(3). (X, d) is said to be b_2-complete if every b_2-Cauchy sequence is a b_2-convergent sequence.

Definition 2.5. *[15] Let (X, d) and (X', d') be two b_2-metric spaces and left $f:X \to X'$ be a mapping.Then f is said to be b_2-continuous at a point $z \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $d(z, x, a) < \delta$ for all $a \in X$ imply that $d'(fz, fx.a) < \varepsilon$. The mapping f is b_2-continuous on X if it is b_2-continuous at all $z \in X$.*

Definition 2.6. *[15] Let (X, d) and (X', d') be two b_2-metric spaces. Then a mapping f: $X \to X'$ is b_2-continuous at a point $x \in X$ if and only if it is b_2-sequentially continuous at x; that is , whenever $\{x_n\}$is b_2-convergent to x, $\{fx_n\}$is b_2-convergent to $f(x)$.*

Definition 2.7. *[16] Let $s \geq 1$ be a constant. A mapping φ: $[0, +\infty) \to [0, +\infty)$ is called comparison function with base $s \geq 1$, if the following two axioms are fulfilled:*
(a) φ is non-decreasing,
(b) $\lim_{n\to+\infty} \varphi^n(t) = 0$ for all $t > 0$.
Clearly, if φ is a comparison function, then $\varphi(t) < t$ for each $t > 0$.

3 Main Results

These are the main results of the paper.

Lemma 3.1. *Let (X, d) be a b_2-metric space with a constant $s > 1$ exist a sequence $\{x_n\}$. Suppose that there is a constant $L < \frac{1}{1+s}$ and a comparison function φ such that the inequality*

$$sd(T_ix, T_jy, a) \leq \varphi(max\{sd(x, T_ix, a), sd(y, T_jy, a), L[d(x, T_jy, a)+d(T_ix, y, a)]\}) \tag{1}$$

holds for each $x, y, a \in X$ and $n \in \mathbb{N}$. Then $\{x_n\}$ is Cauchy sequence.

Proof. For a given point $x_0 \in X$, we inductively define a sequence $\{x_n\}$ by

$$x_{n+1} = T_{n+1}x_n, \quad n \in \mathbb{N}. \tag{2}$$

We claim that

$$d(x_n, x_{n+1}, x_{n+2}) = 0, \text{ for all } n \in \mathbb{N}. \tag{3}$$

From the contraction condition (1), there is

$$\begin{aligned} sd(x_n, x_{n+1}, x_{n+2}) =& sd(T_{n+1}x_n, T_{n+2}x_{n+1}, x_n) \\ \leq& \varphi(\max\{sd(x_n, T_{n+1}x_n, x_n), sd(x_{n+1}, T_{n+2}x_{n+1}, x_n), \\ & L[d(x_n, T_{n+2}x_{n+1}, x_n) + d(T_{n+1}x_n, x_{n+1}, x_n)]\}) \\ =& \varphi(sd(x_{n+2}, x_{n+1}, x_n)). \end{aligned}$$

Suppose that $d(x_{n+1}, x_{n+2}, x_n) > 0$. Since $\varphi(t) < t$ for all $t > 0$, then we have

$$sd(x_n, x_{n+1}, x_{n+2}) \leq \varphi(sd(x_{n+2}, x_{n+1}, x_n) < sd(x_{n+2}, x_{n+1}, x_n).$$

This is a contradiction. Therefore $d(x_n, x_{n+1}, x_{n+2}) = 0$.
We claim that

$$sd(x_n, x_{n+1}, a) \leq \varphi(sd(x_{n-1}, x_n, a)), \text{ for all } a \in X,\ n \in \mathbb{N}. \tag{4}$$

First we have

$$\begin{aligned} sd(x_{n+1}, x_n, a) =& sd(T_{n+1}x_n, T_nx_{n-1}, a) \\ \leq& \varphi(\max\{sd(x_n, T_{n+1}x_n, a), sd(x_{n-1}, T_nx_{n-1}, a), \\ & L[d(x_n, T_nx_{n-1}, a) + d(T_{n+1}x_n, x_{n-1}, a)]\}) \\ =& \varphi(\max\{sd(x_{n+1}, x_n, a), sd(x_{n-1}, x_n, a), \\ & Ld(x_{n+1}, x_{n-1}, a)\}). \end{aligned}$$

Using the triangle inequality and $L < \frac{1}{2}$, we get

$$\begin{aligned} sd(x_{n+1}, x_n, a) \leq& \varphi(\max\{sd(x_n, x_{n+1}, a), sd(x_{n-1}, x_n, a), \\ & Ls[d(x_{n+1}, x_{n-1}, x_n) + d(x_{n-1}, x_n, a) + d(x_{n+1}, x_n, a)]\}) \\ <& \varphi(\max\{sd(x_n, x_{n+1}, a), sd(x_{n-1}, x_n, a), \\ & \frac{s}{2}[d(x_{n+1}, x_n, a) + d(x_{n-1}, x_n, a)]\}) \\ =& \varphi(\max\{sd(x_n, x_{n+1}, a), sd(x_{n-1}, x_n, a)\}). \end{aligned}$$

Suppose that $\max\{sd(x_n, x_{n+1}, a), sd(x_{n-1}, x_n, a)\} = sd(x_n, x_{n+1}, a)$. Then according to the property(a) of φ in Definition 1.7, there is

$$sd(x_{n+1}, x_n, a) < \varphi(sd(x_n, x_{n+1}, a)) < sd(x_n, x_{n+1}, a).$$

which is a contradiction. Thus by the above inequality we have

$$sd(x_{n+1}, x_n, a) \leq \varphi(sd(x_{n-1}, x_n, a)).$$

Hence the inequality (4) holds for all $n \in \mathbb{N}$.
From (4), it is easy to inductively show that

$$sd(x_{n+1}, x_n, a) \leq \varphi^n(sd(x_0, x_1, a)), \text{ for all } a \in X,\ n \in \mathbb{N}. \tag{5}$$

Since $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, from (5) it follows

$$\lim_{n\to\infty} d(x_{n+1}, x_n, a) = 0, \text{ for all } a \in X. \tag{6}$$

Now we go on to show that $\{x_n\}$ is a Cauchy sequence.
Let $\varepsilon > 0$. Since $L < \frac{1}{1+s}$ implies $s - 2L > 0$ and $1 - L(1 + s) > 0$, by (6) we can easily deduce that there exists $n_0 \in \mathbb{N}$ such that

$$d(x_{n-1}, x_n, a) < \frac{1 - L - Ls}{2s}\varepsilon < \varepsilon, \text{ for all } n \geq n_0,\ a \in X. \tag{7}$$

Let $m, n \in \mathbb{N}$ with $m > n$. We claim that

$$d(x_n, x_m, a) < \varepsilon, \text{ for all } m > n \geq n_0,\ a \in X. \tag{8}$$

This is done by induction on m.
Let $n \geq n_0$ and $m = n + 1$. Then from (4) and (7) we get

$$d(x_n, x_m, a) = d(x_n, x_{n+1}, a) < d(x_{n-1}, x_n, a) < \frac{1 - L - Ls}{2s}\varepsilon < \varepsilon.$$

Thus (8) holds for $m = n + 1$.
Assume now that (8) holds for some $m \geq n + 1$. We will show that (8) holds for $m + 1$.
From the contractive condition (1) and (2) there is

$$\begin{aligned} sd(x_n, x_{m+1}, a) =& sd(T_n x_{n-1}, T_{m+1} x_m, a) \\ \leq& \varphi(\max\{sd(x_{n-1}, T_n x_{n-1}, a), sd(x_m, T_{m+1} x_m, a), \\ & L[d(x_{n-1}, T_{m+1} x_m, a) + d(T_n x_{n-1}, x_m, a)]\}) \\ =& \varphi(\max\{sd(x_{n-1}, x_n, a), sd(x_m, x_{m+1}, a), \\ & L[d(x_{n-1}, x_{m+1}, a) + d(x_n, x_m, a)]\}) \\ =& \varphi(\max\{sd(x_{n-1}, x_n, a), L[d(x_{n-1}, x_{m+1}, a) + d(x_n, x_m, a)]\}). \end{aligned}$$

By (4) and $\varphi(t) < t$ for all $t > 0$, then we get

$$sd(x_n, x_{m+1}, a) < \max\{sd(x_{n-1}, x_n, a), L[d(x_{n-1}, x_{m+1}, a) + d(x_n, x_m, a)]\}. \tag{9}$$

If from (9) we have $sd(x_n, x_{m+1}, a) < sd(x_{n-1}, x_n, a)$, then by (7) there is

$$d(x_n, x_{m+1}, a) < d(x_{n-1}, x_n, a) < \frac{1 - L - Ls}{2s}\varepsilon < \varepsilon.$$

If (9) implies $sd(x_n, x_{m+1}, a) < L[d(x_{n-1}, x_{m+1}, a) + d(x_n, x_m, a)]$, then by the triangle inequality, there is

$$sd(x_n, x_{m+1}, a) < L[sd(x_{n-1}, x_n, a)+sd(x_n, x_{m+1}, a)+sd(x_n, x_{n-1}, x_{m+1})+d(x_n, x_m, a)].$$

Now we turn to prove that $d(x_n, x_{n-1}, x_{m+1}) = 0$.
From (3) we have $d(x_n, x_{n+1}, x_{n+2}) = 0$ for all $n \in N$. Thus we can get

$$\begin{aligned} d(x_{n-1}, x_n, x_{n+2}) \leq& s[d(x_{n-1}, x_n, x_{n+1}) + d(x_n, x_{n+2}, x_{n+1}) + d(x_{n-1}, x_{n+2}, x_{n+1})] \\ =& sd(x_{n-1}, x_{n+1}, x_{n+2}) \\ \leq& sd(x_{n-1}, x_n, x_{n+1}) \\ =& 0. \end{aligned}$$

Similarly, we can get $d(x_{n-1}, x_n, x_{m+1}) = 0$.
Thus $sd(x_n, x_{m+1}, a) < L[sd(x_{n-1}, x_n, a) + sd(x_n, x_{m+1}, a) + d(x_n, x_m, a)]$.
Since $L < \frac{1}{1+s}$ implies $\frac{L}{1-L} < 2L < 1 < s$, we get

$$\begin{aligned} d(x_n, x_{m+1}, a) <& \frac{L}{1 - L}[d(x_{n-1}, x_n, a) + \frac{1}{s}d(x_n, x_m, a)] \\ <& 2L[d(x_{n-1}, x_n, a) + \frac{1}{s}d(x_n, x_m, a)]. \end{aligned}$$

Now by (7) and the inductive hypothesis (8), there is

$$\begin{aligned} d(x_n, x_{m+1}, a) <& 2L\frac{1 - L - Ls}{2s}\varepsilon + \frac{2L}{s}\varepsilon \\ <& \frac{1 - 2L - L(s - 1)}{s}\varepsilon + \frac{2L}{s}\varepsilon \\ <& \frac{1 - 2L}{s}\varepsilon + \frac{2L}{s}\varepsilon\varepsilon. \end{aligned}$$

Thus we have proved that (8) holds for $m + 1$.
From (8) it follows that $\{x_n\}$ is a Cauchy sequence. □

By Lemma 3.1, we get the following the fixed point theorem.

Theorem 3.2. *Let (X, d) be a complete b_2-metric space with a constant $s > 1$ and a family of self-mappings on X, written as $\{T_i\}_{i\in\mathbb{N}}$. Suppose that there is a sequence $\{x_n\}$ satisfy Lemma 3.1. Then $\{T_i\}_{i\in\mathbb{N}}$ have a unique common fixed point.*

Proof. By Lemma 3.1, we have $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete b_2 - metric space, then $\{x_n\}$ converges to some $u \in X$ when $n \to \infty$. For any fixed $n \in \mathbb{N}$, we select sufficiently large $m \in \mathbb{N}$ with $m > n$.
Now from the contractive condition (1) and (2), we have

$$\begin{aligned} sd(u, T_n u, a) =& sd(T_{m+1}x_m, T_n u, a) \\ \leq& \varphi(\max\{sd(x_m, T_{m+1}x_m, a), sd(u, T_n u, a), \\ & L[d(x_m, T_n u, a) + d(T_{m+1}x_m, u, a)]\}) \\ =& \varphi(\max\{sd(x_m, x_{m+1}, a), sd(T_n u, u, a), \\ & L[d(x_m, T_n u, a) + d(x_{m+1}, u, a)]\}). \end{aligned}$$

Let $m \to +\infty$, we have $x_m \to u$, thus $sd(u, T_n u, a) \leq \varphi(sd(T_n u, u, a))$. If we suppose that $d(T_n u, u, a) > 0$, then we have

$$sd(u, T_n u, a) \leq \varphi(sd(T_n u, u, a)) < sd(T_n u, u, a).$$

which is a contradiction. Therefore there is $d(T_n u, u, a) = 0$ and hence $u = T_n u$. Thus we have proved that u is the common fixed point of the $\{T_i\}_{i \in \mathbb{N}}$.
Now suppose that u and v are two different common fixed points of $\{T_i\}_{i \in \mathbb{N}}$, from Definition 2.2(1), we have $d(u, v, a) > 0$ where $a \in X$ and $a \neq u, v$. Then

$$\begin{aligned} sd(u, v, a) =& sd(T_1 u, T_2 v, a) \\ \leq& \varphi(\max\{sd(u, T_1 u, a), sd(v, T_2 v, a), L[d(u, T_2 v, a) + d(T_1 u, v, a)]\}) \\ =& \varphi(L[d(u, v, a) + d(u, v, a)]) \\ <& \varphi(sd(u, v, a)). \end{aligned}$$

Thus we have $sd(u, v, a) < \varphi(sd(u, v, a)) < sd(u, v, a)$ which is a contradiction. So, we have proved that $\{T_i\}_{i \in \mathbb{N}}$ have a unique common fixed point in X. □

Example 3.3. *Let $X = \{(\alpha, 0) : \alpha \in [0, +\infty)\} \cup \{(0, 2)\} \subset \mathbb{R}^2$, $d(x, y, z)$ denote the square of the area of triangle with vertices $x, y, z \in X$, e.g.,*
$d((\alpha, 0), (\beta, 0), (0, 2)) = (\alpha - \beta)^2$.

It is easy to check that d is a b_2-metric with parameter $s = 2$. Consider the mappings $\{T_i\}_{i \in N} : X \to X$ given by

$$for\ all\ \alpha \in [0, +\infty),\ T_i(\alpha, 0) = \begin{cases} (\dfrac{\alpha}{4i}, 0), & i \neq 0; \\ (0, 0), & i = 0. \end{cases}$$

$T_i(0, 2) = (0, 0)$, $i \in N$, and $L = \frac{1}{4} < \frac{1}{1+s}$, comparison function $\varphi(t) = \frac{3}{4}t$.

Finally, in order to check the contractive condition, only the case when $x = (\alpha, 0)$, $y = (\beta, 0)$, $a = (0, 2)$ is nontrivial.

Case1, $ij \neq 0$.

$$\begin{aligned}
sd(T_ix, T_jy, a) &= 2d((\frac{\alpha}{4i}, 0), (\frac{\beta}{4j}, 0), (0, 2)) \\
&= 2(\frac{\alpha}{4i} - \frac{\beta}{4j})^2 \\
&\leq \max\{\frac{\alpha^2}{8}, \frac{\beta^2}{8}\} \\
&< \max\{\frac{27}{32}\alpha^2, \frac{27}{32}\beta^2\} \\
&\leq \frac{3}{2}\max\{(\alpha - \frac{\alpha}{4i})^2, (\beta - \frac{\beta}{4j})^2\} \\
&= \frac{3}{4}\max\{2d((\alpha, 0), (\frac{\alpha}{4i}, 0), (0, 2)), 2d((\beta, 0), (\frac{\beta}{4j}, 0), (0, 2))\} \\
&= \varphi(\max\{sd(x, T_ix, a), sd(y, T_jy, a)\}) \\
&\leq \varphi(\max\{sd(x, T_ix, a), sd(y, T_jy, a), L[d(x, T_jy, a) + d(T_ix, y, a)]\}).
\end{aligned}$$

Thus we check that (1) *holds for* $ij \neq 0$.
Case2, $i = 0, \ j \neq 0$.

$$\begin{aligned}
sd(T_ix, T_jy, a) &= 2d((0, 0), (\frac{\beta}{4j}, 0), (0, 2)) \\
&= 2(0 - \frac{\beta}{4j})^2 \\
&\leq \frac{\beta^2}{8} \\
&< \frac{27}{32}\beta^2 = \frac{3}{2}(\beta - \frac{\beta}{4j})^2 \\
&\leq \frac{3}{2}\max\{(\alpha - 0)^2, (\beta - \frac{\beta}{4j})^2\} \\
&= \frac{3}{4}\max\{2d((\alpha, 0), (0, 0), (0, 2)), 2d((\beta, 0), (\frac{\beta}{4j}, 0), (0, 2))\} \\
&= \varphi(\max\{sd(x, T_ix, a), sd(y, T_jy, a)\}) \\
&\leq \varphi(\max\{sd(x, T_ix, a), sd(y, T_jy, a), L[d(x, T_jy, a) + d(T_ix, y, a)]\}).
\end{aligned}$$

Thus we check that (1) *holds for* $i = 0, j \neq 0$.
Case3, $i \neq 0, \ j = 0$. *The proof of* (1) *in this case is similar to one given in Case2.*
Case4, $i = 0, \ j = 0$.

$$\begin{aligned}
sd(T_ix, T_jy, a) &= 2d((0, 0), (0, 0), (0, 2)) = 0 \\
&\leq \varphi(\max\{sd(x, T_ix, a), sd(y, T_jy, a), L[d(x, T_jy, a) + d(T_ix, y, a)]\}).
\end{aligned}$$

Thus we check that (1) holds for $i = 0, j = 0$.

All the conditions of Theorem 3.2 are satisfied and $\{T_i\}_{i\in\mathbb{N}}$ have a unique common fixed point $(0, 0)$.

Theorem 3.4. *Let (X, d) be a complete b_2 - metric space with a constant $s > 1$ and a family of full self-mappings on X, written as $\{T_i\}_{i=0}^{\infty}$. Let $\{m_i\}_{i=0}^{\infty}$ be a family of non-negative integers. Suppose that there is a constant $L < \frac{1}{1+s}$ and a comparison function φ such that the inequality*

$$sd(x, y, a) \leq \varphi(\max\{sd(T_i^{m_i}x, x, a), sd(T_j^{m_j}y, y, a), L[d(x, T_j^{m_j}y, a)+d(T_i^{m_i}x, y, a)]\})$$

holds for all $x, y, a \in X$, $i \neq j$. Suppose that $T_0^{m_0}$ is an identity mapping. Then $\{T_i\}_{i=0}^{\infty}$ have a unique common fixed point.

Proof. Let $S_i = T_i^{m_i}$ for $i \in \mathbb{N}$. Then for all $x, y, a \in X$ and $i \neq j$ we have

$$sd(x, y, a) \leq \varphi(\max\{sd(S_ix, x, a), sd(S_jy, y, a), L[d(x, S_jy, a) + d(S_ix, y, a)]\}) \tag{10}$$

Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{x_n\}_{n\in N}$ by the recursive relation

$$x_{n-1} = S_nx_n, n \in \mathbb{N}. \tag{11}$$

We claim that

$$d(x_n, x_{n+1}, x_{n+2}) = 0, \text{ for all } n \in \mathbb{N}. \tag{12}$$

From the quasi-contractive condition (10) there is

$$\begin{aligned} sd(x_{n+2}, x_{n+1}, x_n) \leq& \varphi(\max\{sd(S_{n+2}x_{n+2}, x_{n+2}, x_n), sd(S_{n+1}x_{n+1}, x_{n+1}, x_n), \\ & L[d(x_{n+2}, S_{n+1}x_{n+1}, x_n) + d(S_{n+2}x_{n+2}, x_{n+1}, x_n)]\}) \\ =& \varphi(\max\{sd(x_{n+1}, x_{n+2}, x_n), sd(x_n, x_{n+1}, x_n), \\ & L[d(x_{n+2}, x_n, x_n) + d(x_{n+1}, x_{n+1}, x_n)]\}) \\ =& \varphi(sd(x_{n+1}, x_{n+2}, x_n)). \end{aligned}$$

Assume that $d(x_{n+1}, x_{n+2}, x_n) > 0$. Since $\varphi(t) < t$ for all $t > 0$, then we have

$$sd(x_n, x_{n+1}, x_{n+2}) \leq \varphi(sd(x_{n+1}, x_{n+2}, x_n)) < sd(x_n, x_{n+1}, x_{n+2}).$$

which is a contradiction. Hence $d(x_n, x_{n+1}, x_{n+2}) = 0$.

Similarly, using the method of Lemma 3.1, we can get that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete b_2-metric space, then $\{x_n\}$ converges to some $x \in X$ when $n \to \infty$. For any fixed $n \in \mathbb{N}$, we select a sufficiently large number $m \in \mathbb{N}$ with $m > n$.

Now, from the contractive condition (11) and (10), there is

$$\begin{aligned}d(x,S_nx,a)\leq&s[d(x,S_nx,x_{m+1})+d(S_nx,a,x_{m+1})+d(x,a,x_{m+1})]\\=&sd(S_nx,x_{m+1},a)\\\leq&\varphi(\max\{sd(S_0(S_nx),S_nx,a),sd(S_{m+1}x_{m+1},x_{m+1},a),\\&L[d(S_nx,S_{m+1}x_{m+1},a)+d(S_0(S_nx),x_{m+1},a)]\})\\=&\varphi(\max\{sd(S_nx,S_nx,a),sd(x_m,x_{m+1},a),\\&L[d(S_nx,x_m,a)+d(S_nx,x_{m+1},a)]\})\\=&\varphi(\max\{sd(x_m,x_{m+1},a),L[d(S_nx,x_m,a)+d(S_nx,x_{m+1},a)]\}).\end{aligned}$$

Let $m\to+\infty$, we have $x_m\to x$, thus $d(x,S_nx,a)\leq\varphi(d(S_nx,x,a))$.

Suppose that $d(S_nx,x,a)>0$, then we have

$$d(x,S_nx,a)\leq\varphi(d(S_nx,x,a))<d(S_nx,x,a)$$

. which is a contradiction. Therefore $d(S_nx,x,a)=0$. Hence $x=S_nx$ for all $n\in\mathbb{N}$. Thus we have proved that x is the common fixed point of the $\{S_i\}_{i=0}^{\infty}$.

Suppose that x and y are two different common fixed points of $\{S_i\}_{i=0}^{\infty}$, from Definition 2.2(1), we know that there exist $a\in X$ and $a\neq u,v$ satisfy $d(x,y,a)>0$. Then there is

$$\begin{aligned}sd(x,y,a)\leq&\varphi(\max\{sd(S_{n+1}x,x,a),sd(S_ny,y,a),\\&L[d(x,S_nya)+d(S_{n+1}x,y,a)]\})\\=&\varphi(\max\{sd(x,x,a),sd(y,y,a),L[d(x,y,a)+d(x,y,a)]\})\\<&\varphi(\frac{1}{2}[d(x,y,a)+d(x,y,a)])\\=&\varphi(d(x,y,a)).\end{aligned}$$

It follows that $sd(x,y,a)<\varphi(d(x,y,a))<d(x,y,a)$ which is a contradiction. Thus $\{S_i\}_{i=0}^{\infty}$ have only a unique common fixed point in X.
Since $x=S_nx=T_n^{m_n}x$ for all $n\in N$, there is

$$T_nx=T_n(T_n^{m_n}x)=T_n^{m_n}(T_nx)=S_n(T_nx).$$

Thus T_nx is a fixed point of S_n for all $n\in\mathbb{N}$. Then for every fixed n and $i\in\mathbb{N}(i\neq n)$, $a\in X$, we have

$$\begin{aligned}sd(T_nx,S_i(T_nx),a)\leq&\varphi(\max\{sd(T_nx,S_n(T_nx),a),sd(S_i(T_nx),S_0(S_i(T_nx)),a),\\&L[d(T_nx,S_0(S_i(T_nx)),a)+d(S_n(T_nx),S_i(T_nx),a)]\})\\=&\varphi(\max\{sd(T_nx,T_nx,a),sd(S_i(T_nx),S_i(T_nx),a),\\&L[d(T_nx,S_i(T_nx),a)+d(T_nx,S_i(T_nx),a)]\})\\<&\varphi(\frac{1}{2}[d(T_nx,S_i(T_nx),a)+d(T_nx,S_i(T_nx),a)])\\=&\varphi(d(T_nx,S_i(T_nx),a))\\<&d(T_nx,S_i(T_nx),a).\end{aligned}$$

which is a contradiction. Thus $S_i(T_nx) = T_nx$ for all $i \in \mathbb{N}$. Therefore, for all $n \in \mathbb{N}$, T_nx is a fixed point of $\{S_i\}_{i=0}^{\infty}$. Since $\{S_i\}_{i=0}^{\infty}$ have a unique common fixed point, therefore $T_nx = x$ for all $n \in \mathbb{N}$.
Suppose that x and z are two different common fixed points of $\{T_i\}_{i=0}^{\infty}$. From Definition 2.2(1), we know that there exist $a \in X$ and $a \neq x, z$ satisfy $d(x, z, a) > 0$. Thus

$$\begin{aligned} sd(x,z,a) \leq & \varphi(\max\{sd(T_{n+1}x,z,a), sd(s_nz,z,a), \\ & L[d(x,T_nz,a) + d(T_{n+1}x,z,a)]\}) \\ = & \varphi(\max\{sd(x,x,a), sd(z,z,a), L[d(x,z,a) + d(x,z,a)]\}) \\ < & \varphi(\frac{1}{2}[d(x,z,a) + d(x,z,a)]) \\ = & \varphi(d(x,z,a)). \end{aligned}$$

It follows that $sd(x, z, a) < \varphi(d(x, z, a)) < d(x, z, a)$ which is a contradiction. Thus $\{T_i\}_{i=0}^{\infty}$ have only a unique common fixed point in X. □

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