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Existence of solutions for fractional q-difference equations with nonlocal and sub-strip type fractional boundary conditions

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Abstract

This paper is concerned with new boundary value problems of nonlinear q-fractional differential equations with nonlocal and sub-strip type fractional boundary conditions. the existence and uniqueness of solutions of the equation are proved by using a generalized coupled point theorem in the space of the continuous functions defined on [0,1], fixed point theorem due to O'Regan and Banach's contraction principle. Finally, the correctness of the conclusion in this paper is verified by some examples.

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1 Introduction

In the recent years, extensive studies on fractional boundary value problems indicate that it is one of the hot topics of the present-day research. specially mathematics and engineering sciences. Many natural phenomena can be present by boundary value problems of fractional differential equations. Many authors in different fields such as chemical physics, engineering, biology, fluid flows, electrical networks, visco-elasticity, try to modeling of these phenomena by boundary value problems of fractional differential equations [1-4]. The early work on q-difference calculus or quantum calculus dates back to Jacksons paper [5]. Basic definitions and properties of quantum calculus can be found in the book [6]. The fractional q-difference calculus had its origin in the works by Al-Salam [7] and Agarwal [8]. Motivated by recent interest in the study of fractional-order differential equations, the topic of q-fractional equations has attracted the attention of many researchers.

In 2012, Bashir Ahmad and Sotiris K Ntouya[9] studied the existence of solutions for a new class of nonlocal boundary value problems of nonlinear differential equations and inclusions of fractional order with strip conditions:

$$\begin{cases} ({}^cD^q x)(t) = f(t, x(t)), 0 < t < 1, 1 < q \leq 2; \\ x(0) = \sigma \int_\alpha^\beta x(s)ds, x(1) = \eta \int_\gamma^\delta x(s)ds, 0 < \alpha < \beta < \gamma < \delta < 1, \end{cases}$$

where ${}^cD^q$ denotes the Caputo fractional derivation of order $q, f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and σ, η are appropriately chosen real numbers.

In 2014, Suphawat Asawasamrit *et al*[10] studied the existence of solutions for nonlocal fractional q-integral boundary value problem of nonlinear fractional q-integrodifference equation:

$$\begin{cases} D_q^\alpha x(t) = f(t, x(t), I_z^\delta x(t)), t \in (0, T); \\ x(0) = 0, \lambda I_p^\beta x(\eta) = I_r^\gamma x(\xi), \end{cases}$$

where $0 < p, q, r, z < 1, 1 < \alpha \leq 2, \beta, \gamma, \delta > 0, \lambda \in \mathbb{R}$ are given constants, D_q^α is the fractional q-derivative of Riemann-Liouville type of order α, I_ϕ^ψ is the fractional ϕ-integral of order ψ with $\phi = p, r, z$, and $\psi = \beta, \gamma, \delta, f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In this paper, we consider the following boundary value problem of fractional q-difference inclusions with nonlocal and sub-strip type boundary conditions:

$$\begin{cases} (D_q^\alpha u)(t) = Mf(t, u(t), (D_q^\mu u)(t)) + NI_q^\beta g(t, u(t), (D_q^\mu u)(t)), t \in [0,1]; \\ u(0) = \int_\lambda^\gamma u(s)d_qs = 0, (D_q^\nu u)(1) = k \int_\xi^\eta u(s)d_qs, \end{cases} \tag{1}$$

where $0 < \lambda < \gamma < \xi < \eta < 1, f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are given continuous functions D_q^α, D_q^μ and D_q^ν denote the fractional q-derivative of Riemane-Liouville type of order $2 < \alpha \leq 3, 0 < \mu, \nu < 1$, $\alpha - \mu > 2, \alpha - \nu > 2$, $I_q^\beta(\cdot)$ denotes Riemann-Liouville integral with $0 < \beta < 1$, k is appropriately chosen real number and M, N being real constants.

The paper is organized as follows. In Section 2, we recall some fundamental concepts of fractional q-calculus and establish a lemma for the linear variant of the given problem. Section 3 contains the existence results for the problem (1) which are shown by applying a generalized coupled point theorem in the space of the continuous functions defined on [0,1], fixed point theorem due to O'Regan and Banach's contraction principle. Finally, we present three examples to illustrate our main results.

2 Preliminary

To make this paper self-contained, below we recall some known facts on fractional q-calculus. The presentation here can be found in([11]-[12]).

For a real parameter $q \in (0,1)$, a q-real number denoted by $[a]_q$ is defined by

$$[a]_q = \frac{1-q^a}{1-q}, a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}_0 := \{0,1,2,...\}$ is defined by

$$(a-b)^0 = 1, (a-b)^{(n)} = \prod_{k=0}^{n-1}(a-bq^k), n \in \mathbb{N}, a,b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$,then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \frac{a-bq^k}{a-bq^{\alpha+k}}, a \neq 0.$$

Clearly, if $b=0$,then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, x \in \mathbb{R}\backslash\{0,-1,-2,...\}.$$

and satisfies $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.
The q-derivative of a function f is defined by

$$(D_qf)(x) = \frac{f(qx)-f(x)}{(q-1)x}, (D_qf)(0) = \lim_{x\to 0}(D_qf)(x).$$

and q-derivative of higher order by

$$(D_q^0 f)(x) = f(x), (D_q^n f)(x) = D_q(D_q^{n-1}f)(x), n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0,b]$ is given by

$$(I_qf)(x) = \int_0^x f(s)d_qs = x(1-q)\sum_{k=0}^{\infty} f(xq^k)q^k, x \in [0,b].$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$,then its integral from a to b is defined by

$$\int_a^b f(s)d_qs = \int_0^b f(s)d_qs - \int_0^a f(s)d_qs.$$

Similar to that for derivatives, an operator I_q^n is given by

$$(I_q^0 f)(x) = f(x), (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, *i.e.*

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Definition 2.1 *Let $\alpha \geq 0$ and f be function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0 f) = f(x)$ and*

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t, \alpha > 0, x \in [0,1].$$

Definition 2.2 *The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined $D_q^0 f(x) = f(x)$ and $(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \alpha > 0$, where m is the smallest integer greater than or equal to α.*

Definition 2.3 *An element $(x, y) \in C[0,1] \times C[0,1]$ is said to a φ-coupled fixed point of a mapping $G : C[0,1] \times C[0,1] \to C[0,1]$ if $G(x, y) = x$ and $G(\overline{x}, \overline{y}) = y$. where $\overline{x} = x(\varphi(t))$ for $t \in [0,1]$, φ is a continuous function.*

Lemma 2.4 [13] *Let $\alpha, \beta \geq 0$, and f be a function defined in [0,1]. Then,the following formulas hold: $(1)(I_q^\beta I_q^\alpha f)(x) = I_q^{\alpha+\beta} f(x)$; $(2)(D_q^\alpha I_q^\alpha f)(x) = f(x)$.*

Lemma 2.5 [13] *Let $\alpha > 0$ and n be a positive integer. Then, the following equality holds:*

$$(I_q^\alpha D_q^n f)(x) = (D_q^n I_q^n f)(x) - \sum_{k=0}^{n-1} \frac{x^{\alpha-n+k}}{\Gamma_q(a + k - n + 1)} (D_q^k f)(0).$$

Lemma 2.6 [13] *Let $\alpha \in \mathbb{R}^+, \lambda \in (-1, +\infty)$, the following is valid:*

$$I_q^\alpha((t - a)^\lambda) = \frac{\Gamma_q(\lambda + 1)}{\Gamma_q(\alpha + \lambda + 1)} (t - a)^{(\alpha+\lambda)}, 0 < a < t < b.$$

Lemma 2.7 *Let $2 < \alpha \leq 3, 0 < \nu < 1, \alpha - \nu > 2, 0 < \lambda < \gamma < \xi < \eta < 1$, and k is appropriately chosen real number. Then for $h \in C[0,1]$, the unique solution of boundary value problem:*

$$\begin{cases} (D_q^\alpha u)(t) = h(t), t \in [0,1]; \\ u(0) = \int_\lambda^\gamma u(s) d_q s = 0; \\ (D_q^\nu u)(1) = k \int_\xi^\eta u(s) d_q s, \end{cases} \qquad (2)$$

is given by

$$u(t)=\int_0^t \frac{(t-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)d_qm$$
$$+\frac{(\delta_1t^{\alpha-2}-\delta_2t^{\alpha-1})}{\Delta}\int_0^1 \frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)}h(m)d_qm$$
$$+\frac{(\delta_3t^{\alpha-2}-\delta_4t^{\alpha-1})}{\Delta}\int_\lambda^\gamma(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)d_qm)d_qs$$
$$+\frac{k(\delta_2t^{\alpha-1}-\delta_1t^{\alpha-2})}{\Delta}\int_\xi^\eta(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)d_qm)d_qs.$$

where $\Delta=\delta_1\delta_4-\delta_2\delta_3\neq 0, \delta_1=\frac{(\gamma^\alpha-\lambda^\alpha)}{[\alpha]_q}, \delta_2=\frac{(\gamma^{\alpha-1}-\lambda^{\alpha-1})}{[\alpha-1]_q}$,
$\delta_3=\frac{k(\eta^\alpha-\xi^\alpha)}{[\alpha]_q}-\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\nu)}, \delta_4=\frac{k(\eta^{\alpha-1}-\xi^{\alpha-1})}{[\alpha-1]_q}-\frac{\Gamma_q(\alpha-1)}{\Gamma_q(\alpha-\nu-1)}$.

Proof It is well known that the solution of q-fractional equation in (2) can be written as

$$u(t)=\int_0^t \frac{(t-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)d_qm+C_1t^{\alpha-1}+C_2t^{\alpha-2}+C_3t^{\alpha-3}. \tag{3}$$

where $C_1, C_2, C_3 \in \mathbb{R}$ are constants. Using the boundary condition $u(0)=0$. we obtain $C_3=0$. by Lemma 2.6, we find that

$$(D_q^\nu u)(t)=\int_0^t \frac{(t-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)}h(m)d_qm+\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\nu)}C_1t^{\alpha-\nu-1}+\frac{\Gamma_q(\alpha-1)}{\Gamma_q(\alpha-\nu-1)}C_2t^{\alpha-\nu-2}.$$

Appling the boundary conditions $\int_\lambda^\gamma u(s)d_qs=0, (D_q^\nu u)(1)=k\int_\xi^\eta u(s)d_qs$ we find that problem:

$$\frac{(\gamma^\alpha-\lambda^\alpha)}{[\alpha]_q}C_1+\frac{(\gamma^{\alpha-1}-\lambda^{\alpha-1})}{[\alpha-1]_q}C_2=-\int_\lambda^\gamma\Big(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)d_qm\Big)d_qs,$$
$$\Big(\frac{k(\eta^\alpha-\xi^\alpha)}{[\alpha]_q}-\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\nu)}\Big)C_1+\Big(\frac{k(\eta^{\alpha-1}-\xi^{\alpha-1})}{[\alpha-1]_q}-\frac{\Gamma_q(\alpha-1)}{\Gamma_q(\alpha-\nu-1)}\Big)C_2$$
$$=\int_0^1 \frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)}h(m)d_qm-k\int_\xi^\eta(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)d_qm)d_qs$$

Solving these equations simultaneously, we obtain

$$C_1=\frac{1}{\Delta}\Big\{-\delta_4\int_\lambda^\gamma(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)d_qm)d_qs$$
$$-\delta_2\int_0^1 \frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)}h(m)d_qm+k\delta_2\int_\xi^\eta(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)d_qm)d_qs\Big\}$$

$$C_2=\frac{1}{\Delta}\Big\{\delta_3\int_\lambda^\gamma(\int_0^s\frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)d_qm)d_qs$$
$$+\delta_1\int_0^1\frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)}h(m)d_qm-k\delta_1\int_\xi^\eta(\int_0^s\frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)d_qm)d_qs\Big\}$$

Substituting the values of C_1 and C_2 in (3), we get the desired result $u(t)$.

Define $B=\{u:u\in C([0,1]),D_q^\mu u\in C([0,1])\}$ equipped with the norm $\|u\|=\max_{t\in[0,1]}|u(t)|+\max_{t\in[0,1]}|(D_q^\mu u)(t)|$, The space B is a Banach space. In this space we consider the classical given by $d(u,v)=\sup_{t\in[0,1]}\{|u(t)-v(t)|\}$, and it is a known fact (B,d) is a complete metric space.

In view of Lemma 2.7, we defined an operator $T:B\to B$ by

$$(Tu)(t)=M\Big[\int_0^t\frac{(t-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(m,u(m),(D_q^\mu u)(m))d_qm$$
$$+\frac{(\delta_1t^{\alpha-2}-\delta_2t^{\alpha-1})}{\Delta}\int_0^1\frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)}f(m,u(m),(D_q^\mu u)(m))d_qm$$
$$+\frac{(\delta_3t^{\alpha-2}-\delta_4t^{\alpha-1})}{\Delta}\int_\lambda^\gamma(\int_0^s\frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(m,u(m),(D_q^\mu u)(m))d_qm)d_qs$$
$$+\frac{k(\delta_2t^{\alpha-1}-\delta_1t^{\alpha-2})}{\Delta}\int_\xi^\eta(\int_0^s\frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(m,u(m),(D_q^\mu u)(m))d_qm)d_qs\Big]$$
$$+N\Big[\int_0^t\frac{(t-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}g(m,u(m),(D_q^\mu u)(m))d_qm$$
$$+\frac{(\delta_1t^{\alpha-2}-\delta_2t^{\alpha-1})}{\Delta}\int_0^1\frac{(1-qm)^{(\alpha+\beta-\nu-1)}}{\Gamma_q(\alpha+\beta-\nu)}g(m,u(m),(D_q^\mu u)(m))d_qm$$
$$+\frac{(\delta_3t^{\alpha-2}-\delta_4t^{\alpha-1})}{\Delta}\int_\lambda^\gamma(\int_0^s\frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}g(m,u(m),(D_q^\mu u)(m))d_qm)d_qs$$
$$+\frac{k(\delta_2t^{\alpha-1}-\delta_1t^{\alpha-2})}{\Delta}\int_\xi^\eta(\int_0^s\frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}g(m,u(m),(D_q^\mu u)(m))d_qm)d_qs\Big]$$

for any $t\in[0,1]$.

For the sake of convenience, we set

$$\Delta_1=\frac{|\delta_1|+|\delta_2|}{|\Delta|};\Delta_2=\frac{|\delta_3|+|\delta_4|}{|\Delta|};$$
$$\Delta_3=\frac{\Gamma_q(\alpha-1)|\delta_1|}{|\Delta|\Gamma_q(\alpha-\mu-1)}+\frac{\Gamma_q(\alpha)|\delta_2|}{|\Delta|\Gamma_q(\alpha-\mu)};$$
$$\Delta_4=\frac{\Gamma_q(\alpha-1)|\delta_3|}{|\Delta|\Gamma_q(\alpha-\mu-1)}+\frac{\Gamma_q(\alpha)|\delta_4|}{|\Delta|\Gamma_q(\alpha-\mu)};$$
$$\mu_1=\frac{\Gamma_q(\alpha-\mu+1)+\Gamma_q(\alpha+1)}{\Gamma_q(\alpha-\mu+1)\Gamma_q(\alpha+1)};\mu_2=\frac{\Gamma_q(\alpha+\beta-\mu+1)+\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta-\mu+1)\Gamma_q(\alpha+\beta+1)};$$

$$
\begin{aligned}
\mu_3 &= \frac{\Delta_1}{\Gamma_q(\alpha-\nu+1)} + \frac{|k|\Delta_1(\eta^{\alpha+1}-\xi^{\alpha+1})}{\Gamma_q(\alpha+2)} + \frac{\Delta_2(\gamma^{\alpha+1}-\lambda^{\alpha+1})}{\Gamma_q(\alpha+2)};\\
\mu_4 &= \frac{\Delta_1}{\Gamma_q(\alpha+\beta-\nu+1)} + \frac{|k|\Delta_1(\eta^{\alpha+\beta+1}-\xi^{\alpha+\beta+1})}{\Gamma_q(\alpha+\beta+2)} + \frac{\Delta_2(\gamma^{\alpha+\beta+1}-\lambda^{\alpha+\beta+1})}{\Gamma_q(\alpha+\beta+2)};\\
\mu_5 &= \frac{\Delta_3}{\Gamma_q(\alpha-\nu+1)} + \frac{|k|\Delta_3(\eta^{\alpha+1}-\xi^{\alpha+1})}{\Gamma_q(\alpha+2)} + \frac{\Delta_4(\gamma^{\alpha+1}-\lambda^{\alpha+1})}{\Gamma_q(\alpha+2)};\\
\mu_6 &= \frac{\Delta_3}{\Gamma_q(\alpha+\beta-\nu+1)} + \frac{|k|\Delta_3(\eta^{\alpha+\beta+1}-\xi^{\alpha+\beta+1})}{\Gamma_q(\alpha+\beta+2)} + \frac{\Delta_4(\gamma^{\alpha+\beta+1}-\lambda^{\alpha+\beta+1})}{\Gamma_q(\alpha+\beta+2)};\\
\Lambda &= |M|(\mu_1+\mu_3+\mu_5)+|N|(\mu_2+\mu_4+\mu_6).
\end{aligned}
$$

3 Main results

Now, we prove the existence of solution of (1) relies on a generalized coupled point theorem in the space of the continuous functions defined on [0,1].

For our study, we need to introduce the class of functions $\mathscr{A}$ defined by those functions $\phi : [0,\infty) \to [0,\infty)$ which are nondecreasing and such that $I-\phi \in \mathscr{B}$. where I denotes the identity mapping on $[0,\infty)$ and $\mathscr{B}$ is the class of function $\psi : [0,\infty) \to [0,\infty)$ which is nondecreasing and satisfies $\psi(t) = 0$ if and only if $t = 0$.

Lemma 3.1 [15] *Let $G : C[0,1]\times C[0,1] \to C[0,1]$ be a mapping satisfying*

$$d(G(u_1,v_1),G(u_2,v_2)) \le \phi(max(d(u_1,u_2),d(v_1,v_2)));$$

for any $u_1,v_1,u_2,v_2 \in C[0,1]$.
Then G has a unique φ-coupled fixed point. where $\phi \in \mathscr{A}$ and $\varphi : [0,1] \to [0,1]$ is a continuous function.

Theorem 3.2 *Assume that:*
(H_1) $f,g \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and f,g satisfies

$$
\begin{aligned}
|f(t,u,D_q^{\mu}u)-f(t,v,D_q^{\mu}v)| &\le \tfrac{\gamma_1}{2}\phi_1(max(|u-v|,|D_q^{\mu}u-D_q^{\mu}v|)),\\
|g(t,u,D_q^{\mu}u)-g(t,v,D_q^{\mu}v)| &\le \tfrac{\gamma_2}{2}\phi_2(max(|u-v|,|D_q^{\mu}u-D_q^{\mu}v|)).
\end{aligned}
$$

for any $t\in[0,1]$ and $u,v,D_q^{\mu}u,D_q^{\mu}v \in \mathbb{R}$;
(H_2) $\phi(max(d(u,v),d(D_q^{\mu}u,D_q^{\mu}v))) = max\{\phi_1(max(d(u,v),d(D_q^{\mu}u,D_q^{\mu}v))),$ $\phi_2(max(d(u,v),d(D_q^{\mu}u,D_q^{\mu}v)))\}$
where $\phi,\phi_1,\phi_2 \in \mathscr{A}$ and

$$0<\gamma_1 \le |M|^{-1}\Big[\frac{1}{\Gamma_q(\alpha+1)}+\mu_3\Big]^{-1}, 0<\gamma_2 \le |N|^{-1}\Big[\frac{1}{\Gamma_q(\alpha+\beta+1)}+\mu_4\Big]^{-1}$$

Then problem (1) has a unique solution on $[0,1]$

Proof we define the operator $\widetilde{T} : B \times B \to B$ such that $\widetilde{T}(u, D_q^\mu u)(t) = (Tu)(t)$, for any $(u, D_q^\mu u) \in C[0,1] \times C[0,1], t \in [0,1]$, by (H_1), we have $\widetilde{T}(u, D_q^\mu u) \in C[0,1]$. Notice that a solution $u \in B$ of problem (1) is a φ-coupled fixed point of the function $\widetilde{T} : B \to B$. where $\varphi : [0,1] \to [0,1]$ is the continuous function satisfying $\varphi(t) = \rho t, 0 < \rho < t$.
We will show that $\widetilde{T}$ satisfies assumption of Lemma 3.1.
In fact, taking into account our assumptions $(H_1)-(H_2)$, for $u, v, D_q^\mu u, D_q^\mu v \in C[0,1]$ and $t \in [0,1]$, we have

$$
\begin{aligned}
&d(\widetilde{T}(u, D_q^\mu u), \widetilde{T}(v, D_q^\mu v)) = \sup_{t\in[0,1]} \Big\{ |\widetilde{T}(u, D_q^\mu u)(t) - \widetilde{T}(v, D_q^\mu v)(t)| \Big\} \\
&\leq |M| \Big[\int_0^1 \frac{(1-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(m, u(m), (D_q^\mu u)(m)) - f(m, v(m), (D_q^\mu v)(m))| d_q m \\
&+\Delta_1 \int_0^1 \frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)} |f(m, u(m), (D_q^\mu u)(m)) - f(m, v(m), (D_q^\mu v)(m))| d_q m \\
&+\Delta_2 \int_\lambda^\gamma \Big(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(m, u(m), (D_q^\mu u)(m)) - f(m, v(m), (D_q^\mu v)(m))| d_q m \Big) d_q s \\
&+|k|\Delta_1 \int_\xi^\eta \Big(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(m, u(m), (D_q^\mu u)(m)) - f(m, v(m), (D_q^\mu v)(m))| d_q m \Big) d_q s \Big] \\
&+|N| \Big[\int_0^1 \frac{(1-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} |g(m, u(m), (D_q^\mu u)(m)) - g(m, v(m), (D_q^\mu v)(m))| d_q m \\
&+\Delta_1 \int_0^1 \frac{(1-qm)^{(\alpha+\beta-\nu-1)}}{\Gamma_q(\alpha+\beta-\nu)} |g(m, u(m), (D_q^\mu u)(m)) - g(m, v(m), (D_q^\mu v)(m))| d_q m \\
&+\Delta_2 \int_\lambda^\gamma \Big(\int_0^s \frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} |g(m, u(m), (D_q^\mu u)(m)) - g(m, v(m), (D_q^\mu v)(m))| d_q m \Big) d_q s \\
&+|k|\Delta_1 \int_\xi^\eta \Big(\int_0^s \frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} |g(m, u(m), (D_q^\mu u)(m)) - g(m, v(m), (D_q^\mu v)(m))| d_q m \Big) d_q s \Big] \\
&\leq |M| \Big[\int_0^1 \frac{(1-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \frac{\gamma_1}{2} \phi_1(max(|u(m)-v(m)|, |(D_q^\mu u)(m) - (D_q^\mu v)(m)|)) d_q m \\
&+\Delta_1 \int_0^1 \frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)} \frac{\gamma_1}{2} \phi_1(max(|u(m)-v(m)|, |(D_q^\mu u)(m) - (D_q^\mu v)(m)|)) d_q m \\
&+\Delta_2 \int_\lambda^\gamma \Big(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \frac{\gamma_1}{2} \phi_1(max(|u(m)-v(m)|, |(D_q^\mu u)(m) - (D_q^\mu v)(m)|)) d_q m \Big) d_q s \\
&+|k|\Delta_1 \int_\xi^\eta \Big(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \frac{\gamma_1}{2} \phi_1(max(|u(m)-v(m)|, |(D_q^\mu u)(m) - (D_q^\mu v)(m)|)) d_q m \Big) d_q s \Big] \\
&+|N| \Big[\int_0^1 \frac{(1-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} \frac{\gamma_2}{2} \phi_2(max(|u(m)-v(m)|, |(D_q^\mu u)(m) - (D_q^\mu v)(m)|)) d_q m
\end{aligned}
$$

$$
\begin{aligned}
&+\Delta_1 \int_0^1 \frac{(1-qm)^{(\alpha+\beta-\nu-1)}}{\Gamma_q(\alpha+\beta-\nu)} \frac{\gamma_2}{2}\phi_2(max(|u(m)-v(m)|,|(D_q^\mu u)(m)-(D_q^\mu v)(m)|))d_qm \\
&+\Delta_2 \int_\lambda^\gamma (\int_0^s \frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} \frac{\gamma_2}{2}\phi_2(max(|u(m)-v(m)|,|(D_q^\mu u)(m)-(D_q^\mu v)(m)|))d_qm)d_qs \\
&+|k|\Delta_1 \int_\xi^\eta (\int_0^s \frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} \frac{\gamma_2}{2}\phi_2(max(|u(m)-v(m)|,|(D_q^\mu u)(m)-(D_q^\mu v)(m)|))d_qm)d_qs\Big] \\
&\leq \frac{\gamma_1}{2}\phi_1(max(d(u,v),d(D_q^\mu u, D_q^\mu v)))|M|\Big[\int_0^1 \frac{(1-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qm \\
&+\Delta_1 \int_0^1 \frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)}d_qm + \Delta_2 \int_\lambda^\gamma (\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qm)d_qs \\
&+|k|\Delta_1 \int_\xi^\eta (\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qm)d_qs\Big] \\
&+\frac{\gamma_2}{2}\phi_2(max(d(u,v),d(D_q^\mu u, D_q^\mu v)))|N|\Big[\int_0^1 \frac{(1-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}d_qm \\
&+\Delta_1 \int_0^1 \frac{(1-qm)^{(\alpha+\beta-\nu-1)}}{\Gamma_q(\alpha+\beta-\nu)}d_qm + \Delta_2 \int_\lambda^\gamma (\int_0^s \frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}d_qm)d_qs \\
&+|k|\Delta_1 \int_\xi^\eta (\int_0^s \frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}d_qm)d_qs\Big] \\
&= \frac{|M|\gamma_1}{2}\phi_1(max(d(u,v),d(D_q^\mu u, D_q^\mu v)))\Big[\frac{1}{\Gamma_q(\alpha+1)}+\mu_3\Big] \\
&+\frac{|N|\gamma_2}{2}\phi_2(max(d(u,v),d((D_q^\mu u), D_q^\mu v)))\Big[\frac{1}{\Gamma_q(\alpha+\beta+1)}+\mu_4\Big] \\
&\leq \phi(max(d(u,v),d(D_q^\mu u, D_q^\mu v)))
\end{aligned}
$$

where ϕ, ϕ_1, ϕ_2 are nondecreasing. Therefore, $\widetilde{T}$ satisfies assumptions of Lemma 3.1, consequently, $\widetilde{T}$ has a unique φ-conpled fixed point. Thus, the proof is complete.

Our next existence result relies on a fixed point theorem due to O'Reganin.

Lemma 3.3 [15] *Let U be an open set in a closed, convex set D of a Banach space X. Assume $0 \in U$. Also assume that $T(\overline{U})$ is bounded and that $T : \overline{U} \to D$ is given by $T = T_1 + T_2$, in which $T_1 = \overline{U} \to X$ is continuous and completely continuous and $T_2 = \overline{U} \to X$ is a nonlinear contraction (i.e, there exists a continuous nondecreasing function $\rho : [0,\infty) \to [0,\infty)$ satisfying $\rho(z) < z$ for $z > 0$ such that $\|T_2(x) - T_2(y)\| \leq \rho(\|x-y\|)$ for all $x, y \in \overline{U}$). Then,either*
(i) T has a fixed point in $u \in \overline{U}$; or
(ii) there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda T(u)$, where $\overline{U}$ and ∂U, respectively, represent the closure and boundary of U on D.

In the sequel, to apply Lemma 3.3, we define $T_i : B \to B, i = 1, 2$ by

$$(T_1u)(t) = M\int_0^t \frac{(t-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(m, u(m), (D_q^\mu u)(m))d_qm$$
$$+N\int_0^t \frac{(t-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} g(m, u(m), (D_q^\mu u)(m))d_qm$$
$$(T_2u)(t) = M\Big[\frac{(\delta_1 t^{\alpha-2} - \delta_2 t^{\alpha-1})}{\Delta}\int_0^1 \frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)} f(m, u(m), (D_q^\mu u)(m))d_qm$$
$$+\frac{(\delta_3 t^{\alpha-2} - \delta_4 t^{\alpha-1})}{\Delta}\int_\lambda^\gamma \Big(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(m, u(m), (D_q^\mu u)(m))d_qm\Big)d_qs$$
$$+\frac{k(\delta_2 t^{\alpha-1} - \delta_1 t^{\alpha-2})}{\Delta}\int_\xi^\eta \Big(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(m, u(m), (D_q^\mu u)(m))d_qm\Big)d_qs\Big]$$
$$+N\Big[\frac{(\delta_1 t^{\alpha-2} - \delta_2 t^{\alpha-1})}{\Delta}\int_0^1 \frac{(1-qm)^{(\alpha+\beta-\nu-1)}}{\Gamma_q(\alpha+\beta-\nu)} g(m, u(m), (D_q^\mu u)(m))d_qm$$
$$+\frac{(\delta_3 t^{\alpha-2} - \delta_4 t^{\alpha-1})}{\Delta}\int_\lambda^\gamma \Big(\int_0^s \frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} g(m, u(m), (D_q^\mu u)(m))d_qm\Big)d_qs$$
$$+\frac{k(\delta_2 t^{\alpha-1} - \delta_1 t^{\alpha-2})}{\Delta}\int_\xi^\eta \Big(\int_0^s \frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} g(m, u(m), (D_q^\mu u)(m))d_qm\Big)d_qs\Big]$$

Clearly

$$(Tu)(t) = (T_1u)(t) + (T_2u)(t), t \in [0, 1]$$

Theorem 3.4 *Assume that*
(H_3) $f, g : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *are continuous functions such that*

$$|f(t, u, D_q^\mu u) - f(t, v, D_q^\mu v)| \le L_1(|u - v| + |D_q^\mu u - D_q^\mu v|),$$
$$|g(t, u, D_q^\mu u) - g(t, v, D_q^\mu v)| \le L_2(|u - v| + |D_q^\mu u - D_q^\mu v|)$$

for all $t \in [0, 1], L_1, L_2 > 0, u, v, D_q^\mu u, D_q^\mu v \in \mathbb{R}$.
(H_4) *there exist functions* $h_1, h_2 \in C([0, 1], \mathbb{R}^+)$, *and nondecreasing functions* $\psi_1, \psi_2 : \mathbb{R}^+ \to \mathbb{R}^+, (i = 1, 2)$, *such that* $|f(t, u, D_q^\mu u)| \le h_1(t)\psi_1(\|u\|), |g(t, u, D_q^\mu u)| \le h_2(t)\psi_2(\|u\|)|$ *for all* $t \in [0, 1], u, D_q^\mu u \in \mathbb{R}$; *where* $\|h_i(t)\| = \max_{0\le t\le 1} |h_i(t)|, i = 1, 2$;
(H_5) $|M|L_1(\mu_3 + \mu_5) + |N|L_2(\mu_4 + \mu_6) < 1$;
(H_6) *there exists a constant* $r > 0$ *such that*

$$\frac{r}{|M|h_1(t)\psi_1(\|r\|)(\mu_1 + \mu_3 + \mu_5) + |N|h_2(t)\psi_2(\|r\|)(\mu_2 + \mu_4 + \mu_6)} > 1$$

Then the boundary value problem (1) has at least one solution on $[0, 1]$.

Proof We shall show that the operators operators T_1 and T_2 satisfy all the conditions of Lemma 3.3 on $[0,1]$.
For the sake of clarity, we split the proof into a number of steps.
Step 1. The operator T_1 is continuous and completely continuous. For a positive number r. Let us consider the set

$$\overline{B}_r=\{u\in B:\|u\|\leq r\}$$

and show that $T_1(\overline{B}_r)$ is bounded. For any $u\in\overline{B}_r$, we have

$$\begin{aligned}&|(T_1u)(t)|=|M|\int_0^t\frac{(t-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(m,u(m),(D_q^{\mu}u)(m))|d_qm\\&+|N|\int_0^t\frac{(t-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}|g(m,u(m),(D_q^{\mu}u)(m))|d_qm\\&\leq\frac{|M|\|h_1\|\psi_1(\|r\|)}{\Gamma_q(\alpha+1)}+\frac{|N|\|h_2\|\psi_2(\|r\|)}{\Gamma_q(\alpha+\beta+1)}\end{aligned}$$

On the other hand we have

$$\begin{aligned}&|D_q^{\mu}(T_1u)(t)|\leq|M|\int_0^t\frac{(t-qm)^{(\alpha-\mu-1)}}{\Gamma_q(\alpha-\mu)}|f(m,u(m),(D_q^{\mu}u)(m))|d_qm\\&+|N|\int_0^t\frac{(t-qm)^{(\alpha+\beta-\mu-1)}}{\Gamma_q(\alpha+\beta-\mu)}|g(m,u(m),(D_q^{\mu}u)(m))|d_qm\\&\leq\frac{|M|\|h_1\|\psi_1(\|r\|)}{\Gamma_q(\alpha-\mu+1)}+\frac{|N|\|h_2\|\psi_2(\|r\|)}{\Gamma_q(\alpha+\beta-\mu+1)}\end{aligned}$$

thus, we have

$$\begin{aligned}&\|T_1u\|\leq\frac{\Gamma_q(\alpha-\mu+1)+\Gamma_q(\alpha+1)}{\Gamma_q(\alpha-\mu+1)\Gamma_q(\alpha+1)}|M|\|h_1\|\psi_1(\|r\|)\\&+\frac{\Gamma_q(\alpha+\beta-\mu+1)+\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta-\mu+1)\Gamma_q(\alpha+\beta+1)}|N|\|h_2\|\psi_2(\|r\|)\\&=|M|\|h_1\|\psi_1(\|r\|)\mu_1+|N|\|h_2\|\psi_2(\|r\|)\mu_2\end{aligned}$$

Thus the operator $T_1(\overline{B}_r)$ is uniformly bounded. For any $t_1,t_2\in[0,1],t_1<t_2$, we have

$$\begin{aligned}&|(T_1u)(t_1)-(T_1u)(t_2)|\leq\frac{|M|}{\Gamma_q(\alpha)}\Big\{\int_0^{t_1}\Big[(t_2-qm)^{(\alpha-1)}-(t_1-qm)^{(\alpha-1)}\Big]\\&\times|f(m,u(m),(D_q^{\mu}u)(m))|d_qm+\int_{t_1}^{t_2}(t_2-qm)^{(\alpha-1)}|f(m,u(m),(D_q^{\mu}u)(m))|d_qm\Big\}\\&+\frac{|N|}{\Gamma_q(\alpha+\beta)}\Big\{\int_0^{t_1}\Big[(t_2-qm)^{(\alpha+\beta-1)}-(t_1-qm)^{(\alpha+\beta-1)}\Big]|g(m,u(m),(D_q^{\mu}u)(m))|d_qm\end{aligned}$$

$$+\int_{t_1}^{t_2}(t_2-qm)^{(\alpha+\beta-1)}|g(m,u(m),(D_q^\mu u)(m))|d_qm\Big\}$$
$$\leq\frac{|M|\|h_1\|\psi_1(\|r\|)}{\Gamma_q(\alpha)}\Big\{\int_0^{t_1}\Big[(t_2-qm)^{(\alpha-1)}-(t_1-qm)^{(\alpha-1)}\Big]d_qm$$
$$+\int_{t_1}^{t_2}(t_2-qm)^{(\alpha-1)}d_qm\Big\}+\frac{|N|\|h_2\|\psi_2(\|r\|)}{\Gamma_q(\alpha+\beta)}\Big\{\int_0^{t_1}\Big[(t_2-qm)^{(\alpha+\beta-1)}$$
$$-(t_1-qm)^{(\alpha+\beta-1)}\Big]d_qm+\int_{t_1}^{t_2}(t_2-qm)^{(\alpha+\beta-1)}d_qm\Big\}$$

On the other hand we have

$$|D_q^\mu(T_1u)(t_1)-D_q^\mu(T_1u)(t_2)|\leq\frac{|M|}{\Gamma_q(\alpha-\mu)}\Big\{\int_0^{t_1}\Big[(t_2-qm)^{(\alpha-\mu-1)}-(t_1-qm)^{(\alpha-\mu-1)}\Big]$$
$$\times|f(m,u(m),(D_q^\mu u)(m))|d_qm+\int_{t_1}^{t_2}(t_2-qm)^{(\alpha-\mu-1)}|f(m,u(m),(D_q^\mu u)(m))|d_qm\Big\}$$
$$+\frac{|N|}{\Gamma_q(\alpha+\beta-\mu)}\Big\{\int_0^{t_1}\Big[(t_2-qm)^{(\alpha+\beta-\mu-1)}-(t_1-qm)^{(\alpha+\beta-\mu-1)}\Big]$$
$$\times|g(m,u(m),(D_q^\mu u)(m))|d_qm+\int_{t_1}^{t_2}(t_2-qm)^{(\alpha+\beta-\mu-1)}|g(m,u(m),(D_q^\mu u)(m))|d_qm\Big\}$$
$$\leq\frac{|M|\|h_1\|\psi_1(\|r\|)}{\Gamma_q(\alpha-\mu)}\Big\{\int_0^{t_1}\Big[(t_2-qm)^{(\alpha-\mu-1)}-(t_1-qm)^{(\alpha-\mu-1)}\Big]d_qm$$
$$+\int_{t_1}^{t_2}(t_2-qm)^{(\alpha-\mu-1)}d_qm\Big\}+\frac{|N|\|h_2\|\psi_2(\|r\|)}{\Gamma_q(\alpha+\beta-\mu)}$$
$$\times\Big\{\int_0^{t_1}\Big[(t_2-qm)^{(\alpha+\beta-\mu-1)}-(t_1-qm)^{(\alpha+\beta-\mu-1)}\Big]d_qm+\int_{t_1}^{t_2}(t_2-qm)^{(\alpha+\beta-\mu-1)}d_qm\Big\}$$

which is independent of u and tends to zero as $t_2-t_1\to 0$. Thus, $T_1(\overline{B}_r)$ is equicontinuous. Hence, by the Arzel-Ascoli theorem, $T_1(\overline{B}_r)$ is a relatively compact set. Now, let $u_n\in\overline{B}_r$ with $\|u_n-u\|\to 0$. Then the limit $\|u_n(t)-u(t)\|\to 0$ is uniformly valid on $[0,1]$. From the uniform continuity of $f(t,u,D_q^\mu u)$ on the compact set $[0,1]\times\overline{B}_r\times\overline{B}_r$. it follows that $\|f(t,u_n(t),(D_q^\mu u_n)(t))-f(t,u(t),(D_q^\mu u)(t))\|\to 0$ is uniformly valid on $[0,1]$. Hence $\|T_1u_n-T_1u\|\to 0$ as $n\to\infty$ which proves the continuity of T_1. This completes the proof of step 1.

Step 2. T_2 is a contraction on $C([0,1],\mathbb{R},\mathbb{R})$. For $u,v\in B$, we have

$$|(T_2u)(t)-(T_2v)(t)|$$
$$\leq|M|\Big[\Delta_1\int_0^1\frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)}|f(m,u(m),(D_q^\mu u)(m))-f(m,v(m),(D_q^\mu v)(m))|d_qm$$
$$+\Delta_2\int_\lambda^\gamma(\int_0^s\frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(m,u(m),(D_q^\mu u)(m))-f(m,v(m),(D_q^\mu v)(m))|d_qm)d_qs$$

$$+|k|\Delta_1\int_\xi^\eta(\int_0^s\frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(m,u(m),(D_q^\mu u)(m))-f(m,v(m),(D_q^\mu v)(m))|d_qm)d_qs\Big]$$
$$+|N|\Big[\Delta_1\int_0^1\frac{(1-qm)^{(\alpha+\beta-\upsilon-1)}}{\Gamma_q(\alpha+\beta-\upsilon)}|g(m,u(m),(D_q^\mu u)(m))-g(m,v(m),(D_q^\mu v)(m))|d_qm$$
$$+\Delta_2\int_\lambda^\gamma(\int_0^s\frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}|g(m,u(m),(D_q^\mu u)(m))-g(m,v(m),(D_q^\mu v)(m))|d_qm)d_qs$$
$$+|k|\Delta_1\int_\xi^\eta(\int_0^s\frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}|g(m,u(m),(D_q^\mu u)(m))-g(m,v(m),(D_q^\mu v)(m))|d_qm)d_qs\Big]$$
$$\leq|M|L_1\|u-v\|\Big[\frac{\Delta_1}{\Gamma_q(\alpha-\upsilon+1)}+\frac{|k|\Delta_1(\eta^{\alpha+1}-\xi^{\alpha+1})}{\Gamma_q(\alpha+2)}+\frac{\Delta_2(\gamma^{\alpha+1}-\lambda^{\alpha+1})}{\Gamma_q(\alpha+2)}\Big]$$
$$+|N|L_2\|u-v\|\Big[\frac{\Delta_1}{\Gamma_q(\alpha+\beta-\upsilon+1)}+\frac{|k|\Delta_1(\eta^{\alpha+\beta+1}-\xi^{\alpha+\beta+1})}{\Gamma_q(\alpha+\beta+2)}+\frac{\Delta_2(\eta^{\gamma+\beta+1}-\xi^{\lambda+\beta+1})}{\Gamma_q(\alpha+\beta+2)}\Big]$$
$$\leq(|M|L_1\mu_3+|N|L_2\mu_4)\|u-v\|$$

On the other hand we have

$$|D_q^\mu(T_2u)(t)-D_q^\mu(T_2v)(t)|\leq(|M|L_1\mu_5+|N|L_2\mu_6)\|u-v\|$$

Thus, we have

$$\|T_2u-T_2v\|\leq[(|M|L_1(\mu_3+\mu_5)+|N|L_2(\mu_4+\mu_6)]\|u-v\|$$

Step 3. The set $T(\overline{B}_r)$ is bounded. For any $u\in\overline{B}_r$, we have

$$|(T_2u)(t)|\leq|M|[\Delta_1\int_0^1\frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)}|f(m,u(m),(D_q^\mu u)(m))|d_qm$$
$$+\Delta_2\int_\lambda^\gamma(\int_0^s\frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(m,u(m),(D_q^\mu u)(m))|d_qm)d_qs$$
$$+|k|\Delta_1\int_\xi^\eta(\int_0^s\frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(m,u(m),(D_q^\mu u)(m))|d_qm)d_qs]$$
$$+|N|[\Delta_1\int_0^1\frac{(1-qm)^{(\alpha+\beta-\nu-1)}}{\Gamma_q(\alpha+\beta-\nu)}|g(m,u(m),(D_q^\mu u)(m))|d_qm$$
$$+\Delta_2\int_\lambda^\gamma(\int_0^s\frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}|g(m,u(m),(D_q^\mu u)(m))|d_qm)d_qs$$
$$+|k|\Delta_1\int_\xi^\eta(\int_0^s\frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}|g(m,u(m),(D_q^\mu u)(m))|d_qm)d_qs]$$
$$\leq|M|\|h_1\|\psi_1(\|r\|)\mu_3+|N|\|h_2\|\psi_2(\|r\|)\mu_4$$

On the other hand we have

$$|D_q^\mu(T_2u)(t)|\leq|M|\|h_1\|\psi_1(\|r\|)\mu_5+|N|\|h_2\|\psi_2(\|r\|)\mu_6$$

Thus, we have

$$\|T_2u\| \le |M|\|h_1\|\psi_1(\|r\|)(\mu_3+\mu_5) + |N|\|h_2\|\psi_2(\|r\|)(\mu_4+\mu_6)$$

Thus, with the boundedness of the set $T_2(\overline{B}_r)$ implies that the set $T(\overline{B}_r)$ is bounded.
Step 4. Finally, it will be shown that either case (i) or case (ii) in Lemma 3.3 holds, we show that the case (ii) is not possible. On the contrary, we suppose that (ii) holds. then, we have that there exist $\lambda \in (0,1)$ and $u \in \partial B_r$, such that $u = \lambda Tu$. so, we have $\|u\| = r$ and

$$\begin{aligned}
u(t) = \lambda M\Big[&\int_0^t \frac{(t-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(m,u(m),(D_q^\mu u)(m))d_qm\\
&+\frac{(\delta_1t^{\alpha-2}-\delta_2t^{\alpha-1})}{\Delta}\int_0^1 \frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)} f(m,u(m),(D_q^\mu u)(m))d_qm\\
&+\frac{(\delta_3t^{\alpha-2}-\delta_4t^{\alpha-1})}{\Delta}\int_\lambda^\gamma\Big(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(m,u(m),(D_q^\mu u)(m))d_qm\Big)d_qs\\
&+\frac{k(\delta_2t^{\alpha-1}-\delta_1t^{\alpha-2})}{\Delta}\int_\xi^\eta\Big(\int_0^s \frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(m,u(m),(D_q^\mu u)(m))d_qm\Big)d_qs\Big]\\
+\lambda N\Big[&\int_0^t \frac{(t-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} g(m,u(m),(D_q^\mu u)(m))d_qm\\
&+\frac{(\delta_1t^{\alpha-2}-\delta_2t^{\alpha-1})}{\Delta}\int_0^1 \frac{(1-qm)^{(\alpha+\beta-\nu-1)}}{\Gamma_q(\alpha+\beta-\nu)} g(m,u(m),(D_q^\mu u)(m))d_qm\\
&+\frac{(\delta_3t^{\alpha-2}-\delta_4t^{\alpha-1})}{\Delta}\int_\lambda^\gamma\Big(\int_0^s \frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} g(m,u(m),(D_q^\mu u)(m))d_qm\Big)d_qs\\
&+\frac{k(\delta_2t^{\alpha-1}-\delta_1t^{\alpha-2})}{\Delta}\int_\xi^\eta\Big(\int_0^s \frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} g(m,u(m),(D_q^\mu u)(m))d_qm\Big)d_qs\Big]
\end{aligned}$$

Using the assumptions (H_4) and (H_6), we get

$$|u(t)| \le \lambda|M|\|h_1\|\psi_1(\|r\|)\Big(\frac{1}{\Gamma_q(\alpha+1)}+\mu_3\Big) + \lambda|N|\|h_2\|\psi_2(\|r\|)\Big(\frac{1}{\Gamma_q(\alpha+\beta+1)}+\mu_4\Big)$$

On the other hand we have

$$\begin{aligned}
|D_q^\mu u(t)| &\le \lambda|M|\|h_1\|\psi_1(\|r\|)\Big(\frac{1}{\Gamma_q(\alpha-\mu+1)}+\mu_5\Big)\\
&+\lambda|N|\|h_2\|\psi_2(\|r\|)\Big(\frac{1}{\Gamma_q(\alpha+\beta-\mu+1)}+\mu_6\Big)
\end{aligned}$$

Thus, we have

$$\|u(t)\| \le \lambda|M|\|h_1\|\psi_1(\|r\|)\Big(\frac{1}{\Gamma_q(\alpha+1)}+\mu_3\Big) + \lambda|N|\|h_2\|\psi_2(\|r\|)\Big(\frac{1}{\Gamma_q(\alpha+\beta+1)}+\mu_4\Big)$$

$$+\lambda|M|\|h_1\|\psi_1(\|r\|)\Big(\frac{1}{\Gamma_q(\alpha-\mu+1)}+\mu_5\Big)+\lambda|N|\|h_2\|\psi_2(\|r\|)\Big(\frac{1}{\Gamma_q(\alpha+\beta-\mu+1)}+\mu_6\Big)$$
$$=\lambda|M|\|h_1\|\psi_1(\|r\|)(\mu_1+\mu_3+\mu_5)+\lambda|N|\|h_2\|\psi_2(\|r\|)(\mu_2+\mu_4+\mu_6)$$

Which yields

$$r\le\lambda|M|\|h_1\|\psi_1(\|r\|)(\mu_1+\mu_3+\mu_5)+\lambda|N|\|h_2\|\psi_2(\|r\|)(\mu_2+\mu_4+\mu_6)$$

Thus, we get a contradiction: Which yields

$$\frac{r}{|M|\|h_1\|\psi_1(\|r\|)(\mu_1+\mu_3+\mu_5)+|N|\|h_2\|\psi_2(\|r\|)(\mu_2+\mu_4+\mu_6)}\le\lambda<1$$

Thus the operators T_1 and T_2 satisfy all the conditions of Lemma 3.3. Hence, the operator T has at least one fixed point $u\in\overline{B}_r$, which is the solution of the problem. This completes the proof.

Theorem 3.5 *Suppose that the assumption* (H_3) *holds and that* $\Lambda K<1$ *and* $L=max\{L_1,L_2\}$. *Then the boundary value problem (1) has a unique solution.*

Proof Let us fix $\max_{t\in[0,1]}|f(t,0,0)|=K_1, \max_{t\in[0,1]}|g(t,0,0)|=K_2$ and $K=max\{K_1,K_2\}$, choosing $r>\frac{\overline{\Lambda}}{1-\Lambda L}$, We show that $TB_r\in B_r$, where $\overline{\Lambda}=|M|K_1(\mu_1+\mu_3+\mu_5)+|M|K_2(\mu_2+\mu_4+\mu_6)$. and $B_r=\{u\in B:\|u\|\le r\}$. For $u\in B_r$, we have

$$|(Tu)(t)|\le|M|\Big[\int_0^1\frac{(t-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}[|f(m,u(m),(D_q^\mu u)(m))-f(m,0,0)|+|f(m,0,0)|]d_qm$$
$$+\Delta_1\int_0^1\frac{(1-qm)^{(\alpha-\nu-1)}}{\Gamma_q(\alpha-\nu)}[|f(m,u(m),(D_q^\mu u)(m))-f(m,0,0)|+|f(m,0,0)|]d_qm$$
$$+\Delta_2\int_\lambda^\gamma(\int_0^s\frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}[|f(m,u(m),(D_q^\mu u)(m))-f(m,0,0)|+|f(m,0,0)|]d_qm)d_qs$$
$$+|k|\Delta_1\int_\xi^\eta(\int_0^s\frac{(s-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}[|f(m,u(m),(D_q^\mu u)(m))-f(m,0,0)|+|f(m,0,0)|]d_qm)d_qs\Big]$$
$$+|N|\Big[\int_0^t\frac{(t-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}[|g(m,u(m),(D_q^\mu u)(m))-g(m,0,0)|+|g(m,0,0)|]d_qm$$
$$+\Delta_1\int_0^1\frac{(1-qm)^{(\alpha+\beta-\nu-1)}}{\Gamma_q(\alpha+\beta-\nu)}[|g(m,u(m),(D_q^\mu u)(m))-g(m,0,0)|+|g(m,0,0)|]d_qm$$
$$+\Delta_2\int_\lambda^\gamma(\int_0^s\frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}[|g(m,u(m),(D_q^\mu u)(m))-g(m,0,0)|+|g(m,0,0)|]d_qm)d_qs$$
$$+|k|\Delta_1\int_\xi^\eta(\int_0^s\frac{(s-qm)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}[|g(m,u(m),(D_q^\mu u)(m))-g(m,0,0)|+|g(m,0,0)|]d_qm)d_qs\Big]$$
$$\le|M|(L_1r+K_1)\Big(\frac{1}{\Gamma_q(\alpha+1)}+\mu_3\Big)+|N|(L_2r+K_2)\Big(\frac{1}{\Gamma_q(\alpha+\beta+1)}+\mu_4\Big)$$

On the other hand we have

$$|D_q^{\mu}(Tu)(t)| \leq |M|(L_1 r + K_1)\Big(\frac{1}{\Gamma_q(\alpha-\mu+1)} + \mu_5\Big) + |N|(L_2 r + K_2)\Big(\frac{1}{\Gamma_q(\alpha+\beta-\mu+1)} + \mu_6\Big)$$

thus, we have

$$\begin{aligned}\|Tu\| &\leq \Big[|M|(\mu_1+\mu_3+\mu_5) + |N|(\mu_2+\mu_4+\mu_6)\Big]Lr \\ &+\Big[|M|K_1(\mu_1+\mu_3+\mu_5) + |N|K_2(\mu_2+\mu_4+\mu_6)\Big] \\ &= \Lambda L r + \overline{\Lambda} \leq r\end{aligned}$$

Which means that $TB_r \in B_r$.
Now, for $u, v \in B$ we obtain

$$|(Tu)(t) - (Tv)(t)| \leq |M|L_1\|u-v\|\Big(\frac{1}{\Gamma_q(\alpha+1)} + \mu_3\Big) + |N|L_2\|u-v\|\Big(\frac{1}{\Gamma_q(\alpha+\beta+1)} + \mu_4\Big)$$

On the other hand we have

$$\begin{aligned}|D_q^{\mu}(Tu)(t) - D_q^{\mu}(Tv)(t)| &\leq |M|L_1\|u-v\|\Big(\frac{1}{\Gamma_q(\alpha-\mu+1)} + \mu_5\Big) \\ &+|N|L_2\|u-v\|\Big(\frac{1}{\Gamma_q(\alpha+\beta-\mu+1)} + \mu_6\Big)\end{aligned}$$

thus, we have

$$\|Tu - Tv\| \leq |M|L_1(\mu_1+\mu_3+\mu_5)\|u-v\| + |N|L_2(\mu_2+\mu_4+\mu_6)\|u-v\| \leq \Lambda L\|u-v\|$$

As $\Lambda L < 1$, therefore $T$is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (the Banach fixed point theorem).

4 Some examples

In this section we present some examples to illustrate our results.

Example 4.1 *Consider the following fractional q-difference boundary value problem:*

$$\begin{cases} (D_q^{2.5}u)(t) = \frac{1}{6}f(t, u(t), (D_q^{0.2}u)(t)) + \frac{1}{4}I_q^{0.5}g(t, u(t), (D_q^{0.2}u)(t)); \\ u(0) = \int_{\frac{1}{6}}^{\frac{1}{4}} u(s)d_qs = 0, (D_q^{0.3}u)(1) = \int_{\frac{1}{3}}^{\frac{1}{2}} u(s)d_qs. \end{cases}$$

where $\alpha = 2.5, \beta = 0.5, \mu = 0.2, \upsilon = 0.3, q = 0.5, \lambda = \frac{1}{6}, \gamma = \frac{1}{4}, \xi = \frac{1}{3}, \eta = \frac{1}{2}, k = 1, M = \frac{1}{6}, N = \frac{1}{4}, f(t, u, D_q^{0.2}u) = \frac{t^2}{2}\Big(\frac{|u(t)|+|(D_q^{0.2}u)(t)|}{1+|u(t)|+|(D_q^{0.2}u)(t)|} + sint\Big), g(t, u, D_q^{0.2}u) =$

$|u(t)|+|(D_q^{0.2}u)(t)|+cost+1$, *It is clear that* $f, g \in C([0,1]\times\mathbb{R}\times\mathbb{R}.\mathbb{R})$. *Moreover for* $t \in [0,1]$ *and* $u, v, D_q^{0.2}u, D_q^{0.2}v \in \mathbb{R}$, *we have*

$$
\begin{aligned}
&|f(t,u,D_q^{0.2}u) - f(t,v,D_q^{0.2}v)| \\
&\leq \frac{1}{2}\Big(\frac{|u(t)-v(t)| + |(D_q^{0.2}u)(t) - (D_q^{0.2}v)(t)|}{(1+|u(t)|+|(D_q^{0.2}u)(t)|)(1+|v(t)|+|(D_q^{0.2}v)(t)|)}\Big) \\
&\leq \frac{1}{2}\Big(\frac{|u(t)-v(t)|}{1+|u(t)-v(t)|} + \frac{|(D_q^{0.2}u)(t) - (D_q^{0.2}v)(t)|}{1+|(D_q^{0.2}u)(t) - (D_q^{0.2}v)(t)|}\Big) \\
&\leq max(\phi_1(|u(t)-v(t)|), \phi_1(|(D_q^{0.2}u)(t) - (D_q^{0.2}v)(t)|)) \\
&= \phi_1(max(|u(t)-v(t)|, |(D_q^{0.2}u)(t) - (D_q^{0.2}v)(t)|))
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
&|g(t,u,D_q^{0.2}u) - g(t,v,D_q^{0.2}v)| \\
&\leq |u(t)-v(t)| + |(D_q^{0.2}u)(t) - (D_q^{0.2}v)(t)| \\
&\leq 2max(\phi_2(|u(t)-v(t)|), \phi_2(|(D_q^{0.2}u)(t) - (D_q^{0.2}v)(t)|)) \\
&= \phi_2(max(\frac{1}{2}|u(t)-v(t)|, \frac{1}{2}|(D_q^{0.2}u)(t) - (D_q^{0.2}v)(t)|))
\end{aligned}
$$

Where $\phi, \phi_1, \phi_2 : [0,\infty) \to [0,\infty)$ *and given by* $\phi_1(t) = \frac{t}{1+t}, \phi_2(t) = \frac{t}{2}$, *so* $\phi(t) = \frac{t}{1+t}$. *It is easily checked that* ϕ *is nondecreasing and* $\phi \in \mathscr{A}$. *With the given data, it is found that* $2 = \gamma_1 < |M|^{-1}\Big[\frac{1}{\Gamma_q(\alpha+1)} + \mu_3\Big]^{-1} \approx 2.477779; 2 = \gamma_2 < |N|^{-1}\Big[\frac{1}{\Gamma_q(\alpha+\beta+1)} + \mu_4\Big]^{-1} \approx 2.602678$. *Thus, By Theorem 3.2, the boundary value problem has a unique solution on* $[0,1]$.

Example 4.2 *Consider the following fractional q-difference boundary value problem:*

$$
\begin{cases}
(D_q^{2.5}u)(t) = \frac{1}{6}f(t,u(t),(D_q^{0.3}u)(t)) + \frac{1}{8}I_q^{0.5}g(t,u(t),(D_q^{0.3}u)(t)); \\
u(0) = \int_{\frac{1}{6}}^{\frac{1}{4}} u(s)d_qs = 0, (D_q^{0.3}u)(1) = \int_{\frac{1}{3}}^{\frac{1}{2}} u(s)d_qs.
\end{cases}
$$

where $\alpha = 2.5, \beta = 0.5, \mu = \upsilon = 0.3, q = 0.5, \lambda = \frac{1}{6}, \gamma = \frac{1}{4}, \xi = \frac{1}{3}, \eta = \frac{1}{2}, k = 1, M = \frac{1}{6}, N = \frac{1}{8}, f(t,u,D_q^{0.3}u) = \frac{1}{(3+t)^2}\Big(|u(t)|+2|(D_q^{0.3}u)(t)|+\frac{|u(t)|}{1+|u(t)|}\Big), g(t,u,D_q^{0.3}u) = \frac{1}{5+t^2}\Big(2|u(t)|+|(D_q^{0.3}u)(t)|+\frac{|(D_q^{0.3}u)(t)|}{1+|(D_q^{0.3}u)(t)|}\Big)$,*It is clear that* $L_1 = \frac{2}{9}; L_2 = \frac{2}{5}; h_1(t) = \frac{1}{(3+t)^2}; h_2(t) = \frac{1}{5+t^2}; \Psi_1(\|u\|) = \Psi_2(\|u\|) = 2\|u\| + 1$. *With the given data, it is found that* $|M|L_1(\mu_3+\mu_5) + |N|L_2(\mu_4+\mu_6) \approx 0.168965 < 1, r > 0.210682$. *Thus all the assumptions of theorem 3.2 are satisfied. Hence, the boundary value problem has at least one solution on* $[0,1]$.

Example 4.3 *Consider the following fractional q-difference boundary value problem:*

$$\begin{cases} (D_q^{2.5}u)(t) = \frac{1}{5}f(t,u(t),(D_q^{0.3}u)(t)) + \frac{1}{10}I_q^{0.5}g(t,u(t),(D_q^{0.3}u)(t)); \\ u(0) = \int_{\frac{1}{6}}^{\frac{1}{4}} u(s)d_qs = 0, (D_q^{0.3}u)(1) = \int_{\frac{1}{3}}^{\frac{1}{2}} u(s)d_qs. \end{cases}$$

where $\alpha = 2.5, \beta = 0.5, \mu = \upsilon = 0.3, q = 0.5, \lambda = \frac{1}{6}, \gamma = \frac{1}{4}, \xi = \frac{1}{3}, \eta = \frac{1}{2}, k = 1, M = \frac{1}{5}, N = \frac{1}{10}, f(t,u,D_q^{0.3}u) = \frac{1}{(t+2)^2}\Big(|u(t)| + \frac{|(D_q^{0.3}u)(t)|}{1+|D_q^{0.3}u)(t)|}\Big) + sint, g(t,u,D_q^{0.3}u) = \frac{t^2}{5}\Big(\frac{|u(t)|}{1+|u(t)|} + |(D_q^{0.3}u)(t)|\Big) + cost.$ *It is easily checked that* $L_1 = \frac{1}{4}, L_2 = \frac{1}{5}$*.as* $|f(t,u,D_q^{0.3}u) - f(t,v,D_q^{0.3}v)| < \frac{1}{4}\Big(|u(t)-v(t)| + |(D_q^{0.3}u)(t) - (D_q^{0.3}v)(t)|\Big); |g(t,u,D_q^{0.3}u) - g(t,v,D_q^{0.3}v)| < \frac{1}{5}\Big(|u(t)-v(t)| + |(D_q^{0.3}u)(t) - (D_q^{0.3}v)(t)|\Big)$, *Clearly* $L = max\{L_1, L_2\} = \frac{1}{4}$, *With the given data, it is found that* $\mu_1 \approx 1.373854; \mu_2 \approx 0.835951; \mu_3 \approx 1.64881; \mu_4 \approx 1.155926; \mu_5 \approx 1.174139; \mu_6 \approx 0.828387; \Delta \approx 1.121388.$ *so* $\Delta L = 0.280347 < 1$. *Which satisfies theorem 3.5. Thus the boundary value problem has a unique solution.*

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