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Integrability for Solutions to Some Nonhomogeneous Quasilinear Elliptic Problems

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Abstract

In this paper we prove an estimate for the measure of superlevel sets for weak solutions u of nonhomogeneous quasilinear elliptic systems

$$-\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\alpha\beta}(x, u(x)) D_j u^\beta(x) \right) = -\sum_{i=1}^{n} D_i f_i^\alpha(x, u(x)), \quad (*)$$

$$\alpha = 1, 2, \cdots, N.$$

The diagonal coefficients $a_{ij}^{\gamma\gamma}(x, y)$ are elliptic for large values of u, the off-diagonal coefficients are small when $|u|$ is large, the faster off-diagonal coefficients decay, the higher integrability of u becomes.

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1 Introduction

Let Ω be a bounged open subset of $R^n, n \geq 3$. For $N \geq 2$, let $a_{ij}^{\alpha\beta} : \Omega \times R^N \to R$ be Carathéodory functions, that is, $a_{ij}^{\alpha\beta}(x, y)$ are mesurable with respect to x and continuous with respect to y. Moreover, they are bounded and elliptic.

In this paper we deal with regularity for weak solutions $u : \Omega \subset R^N \to R^N$ of nonhomogeneous quasilinear elliptic systems

$$-\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\alpha\beta}(x, u(x)) D_j u^\beta(x) \right) = -\sum_{i=1}^{n} D_i f_i^\alpha(x, u(x)), \;\; \alpha = 1, 2, \cdots, N. \tag{1.1}$$

Where $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2, \cdots, n$ and we denote $D = (D_1, D_2, \cdots, D_n)$ to be the gradient operator.

In order to get regularity, we need additional assumptions on the coefficients. If $a_{ij}^{\gamma\beta}(x, y)$ are diagonal

$$a_{ij}^{\gamma\beta}(x, y) = 0 \;\; for \;\; \beta \neq \gamma, \tag{1.2}$$

then the N equations (1.1) are decoupled and maximum principle applies to every component u^γ of $u = (u^1, u^2, \cdots, u^N)$:

$$\sup_{\Omega} u^\gamma \leq \sup_{\partial\Omega} u^\gamma. \tag{1.3}$$

Now we no longer assume that off-diagonal coefficients vanish, we only know that they are small when $|y^\gamma|$ is large: there exist $c_1, c_2, q \in (0, +\infty)$ such that

$$\left| a_{ij}^{\gamma\beta}(x, y) \right| \leq \frac{c_1}{(1 + |y^\gamma|)^q} \;\; for \;\; \beta \neq \gamma, \tag{1.4}$$

$$|f_i^\gamma(x, y)| \leq \frac{c_2}{(1 + |y^\gamma|)^q}. \tag{1.5}$$

We assume ellipticity only for diagonal coefficients $a_{ij}^{\gamma\gamma}(x, y)$ and only for large values of $|y^\gamma|$:

$$0 < \theta \leq |y^\gamma| \;\; \Rightarrow \;\; \nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x, y) \xi_j \xi_i \tag{1.6}$$

for some constants $\theta \in [0, +\infty)$ and $\nu \in (0, +\infty)$. And also diagonal coefficients are assumed to be bounded: there exists $c_3 \in (0, +\infty)$ such that

$$|a_{ij}^{\gamma\gamma}(x, u)| \leq c_3. \tag{1.7}$$

for almost every $x \in \Omega$, for every $y \in R^N$, for all $i, j \in \{1, \ldots, n\}$, for any $\gamma \in \{1, \ldots, N\}$. And we note that both diagonal and off-diagonal coefficients are bounded.

In this paper, the Sobolev space $W^{1,2}(\Omega)$ is defined, as usual, by

$$W^{1.2}(\Omega) = \left\{ v \in L^2(\Omega) : \; D_i v \in L^2(\Omega), i = 1, 2, \cdots, n \right\}.$$

The closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,2}(\Omega)$ is denoted by $W_0^{1,2}(\Omega)$.

The main result of this paper is the following theorem.

Theorem 1.1 *Under assumptions (1.4)-(1.7), let* $u = (u^1, u^2, \cdots, u^N)$ *be a weak solution of the system (1.1), that is,* $u \in W^{1,2}(\Omega, R^N)$ *and*

$$\int_\Omega \sum_{j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x, u(x)) D_j u^\beta(x) D_i v^\alpha(x) dx = \int_\Omega \sum_{i=1}^{n} \sum_{\alpha=1}^{N} f_i^\alpha(x, u(x)) D_i v^\alpha(x) dx, \tag{1.8}$$

holds true for all $v \in W_0^{1,2}(\Omega, R^N)$. *Then*

$$u \in L_{loc}^{2^*(q+1)}(\Omega, R^N),$$

where 2^* *is the Sobolev exponent* $\frac{2n}{n-2}$ *and* $n \geq 3$.

2 Proof of Theorem 1.1

We start as in the proof of theorem 2.1 in [1]. Let $\phi : [0, +\infty)$ be increasing and $C^1([0, +\infty))$. Moreover, we assume that there exists a constant $\tilde{c} \in [1, +\infty)$ such that

$$0 \leq \phi(t) \leq \tilde{c} \quad \forall t \in [0, +\infty), \tag{2.1}$$

$$0 \leq \phi'(t) \leq \tilde{c} \quad \forall t \in [0, +\infty), \tag{2.2}$$

$$0 \leq \phi'(t)t \leq \tilde{c} \quad \forall t \in [0, +\infty). \tag{2.3}$$

Let $B_\rho = B(x_0, \rho)$ and $B_R = B(x_0, R)$ be open balls with the same center x_0 and radii $0 < \rho < R \leq 1$, with $\overline{B_R} \subset \Omega$, We assume that $\eta : R^n \to R$, $\eta \in C_0^1(B_R)$ with $0 \leq \eta \leq 1$ in R^n, $\eta = 1$ on B_ρ, $|D\eta| \leq \frac{2}{R-\rho}$ in R^n. We note that $0 < R - \rho < R \leq 1$, so $\frac{2}{R-\rho} > 2$. We fix $\gamma \in \{1, 2, \cdots, N\}$, we consider the test function $v = (v^1, v^2, \cdots, v^N)$ defined as follows

$$v^\alpha = \begin{cases} 0 & if\ \alpha \neq \gamma, \\ \phi(|u^\alpha|)u^\alpha \eta^2 & if\ \alpha = \gamma. \end{cases} \tag{2.4}$$

It is easy to see that

$$v \in W_0^{1,2}(B_R, R^N) \subset W_0^{1,2}(\Omega, R^N), \tag{2.5}$$

and

$$D_i v^\gamma = [\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)] (D_i u^\gamma)\eta^2 + [\phi(|u^\gamma|)u^\gamma] D_i(\eta^2). \tag{2.6}$$

We insert such a test function v into (1.8), then we can obtain

$$\begin{aligned}
&\int_{\{\theta\le|u^\gamma|\}}\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)D_ju^\gamma[\phi'(|u^\gamma|)|u^\gamma|+\phi(|u^\gamma|)](D_iu^\gamma)\eta^2dx\\
=\;&-\int_{\{\theta>|u^\gamma|\}}\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)D_ju^\gamma[\phi'(|u^\gamma|)|u^\gamma|+\phi(|u^\gamma|)](D_iu^\gamma)\eta^2dx\\
&-\int_\Omega\sum_{i,j=1}^{n}\sum_{\beta\ne\gamma}a_{ij}^{\gamma\beta}(x,u)D_ju^\beta[\phi'(|u^\gamma|)|u^\gamma|+\phi(|u^\gamma|)](D_iu^\gamma)\eta^2dx\\
&-\int_\Omega\sum_{i,j=1}^{n}a_{ij}^{\gamma\gamma}(x,u)D_ju^\gamma\phi(|u^\gamma|)u^\gamma D_i(\eta^2)dx\\
&-\int_\Omega\sum_{i,j=1}^{n}\sum_{\beta\ne\gamma}a_{ij}^{\gamma\beta}(x,u)D_ju^\beta\phi(|u^\gamma|)u^\gamma D_i(\eta^2)dx\\
&+\int_\Omega\sum_{i=1}^{n}f_i^\gamma(x,u)[\phi'(|u^\gamma|)|u^\gamma|+\phi(|u^\gamma|)](D_iu^\gamma)\eta^2dx\\
&+\int_\Omega\sum_{i=1}^{n}f_i^\gamma(x,u)\phi(|u^\gamma|)u^\gamma D_i(\eta^2)dx.
\end{aligned}$$

Now we use ellipticity (1.6) on the left-side and (1.4), (1.5), (1.7) on the right-hand side, we get

$$\begin{aligned}
&\nu\int_{\{\theta\le|u^\gamma|\}}[\phi'(|u^\gamma|)|u^\gamma|+\phi(|u^\gamma|)](Du^\gamma)^2\eta^2dx\\
\le\;& nc_3\int_{\{\theta>|u^\gamma|\}}[\phi'(|u^\gamma|)|u^\gamma|+\phi(|u^\gamma|)]|Du^\gamma|^2\eta^2dx\\
&+\frac{n^2Nc_1}{(1+|u^\gamma|)^q}\int_\Omega|Du|[\phi'(|u^\gamma|)|u^\gamma|+\phi(|u^\gamma|)]|Du^\gamma|\eta^2dx\\
&+nc_3\int_\Omega2\eta|Du^\gamma|\phi'(|u^\gamma|)|u^\gamma||D\eta|dx+\frac{nc_2}{(1+|u^\gamma|)^q}\int_\Omega2\eta\phi(|u^\gamma|)|u^\gamma||D\eta|dx\\
&+\frac{n^2Nc_1}{(1+|u^\gamma|)^q}\int_\Omega|Du|\phi(|u^\gamma|)|u^\gamma||2\eta||D\eta|dx\\
&+\frac{nc_2}{(1+|u^\gamma|)^q}\int_\Omega[\phi'(|u^\gamma|)|u^\gamma|+\phi(|u^\gamma|)]|Du^\gamma|\eta^2dx.
\end{aligned}$$

We add to both sides

$$\nu\int_{\{\theta>|u^\gamma|\}}[\phi'(|u^\gamma|)|u^\gamma|+\phi(|u^\gamma|)](Du^\gamma)^2\eta^2dx,$$

and we get

$$\begin{aligned}
& \nu \int_\Omega [\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)](Du^\gamma)^2\eta^2 dx \\
\leq\ & (\nu + nc_3)\int_{\{\theta > |u^\gamma|\}} [\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|^2\eta^2 dx \\
& + \frac{n^2 N c_4}{(1+|u^\gamma|)^q}\int_\Omega (1+|Du|)[\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|\eta^2 dx \\
& + nc_3 \int_\Omega 2\eta |Du^\gamma|\phi(|u^\gamma|)|u^\gamma||D\eta| dx \\
& + \frac{n^2 N c_5}{(1+|u^\gamma|)^q}\int_\Omega (1+|Du|)\phi(|u^\gamma|)|u^\gamma||2\eta||D\eta| dx.
\end{aligned}$$

We use the inequality $2AB \leq \varepsilon A^2 + B^2/\varepsilon$, then we obtain

$$\begin{aligned}
& \frac{\nu}{2}\int_\Omega [\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)](Du^\gamma)^2\eta^2 dx \\
\leq\ & (\nu + nc_3)\int_{\{\theta > |u^\gamma|\}} [\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)]|Du^\gamma|^2\eta^2 dx \\
& + \left(1 + \frac{4n^2c_3^2}{\nu}\right)\int_\Omega \phi(|u^\gamma|)|u^\gamma|^2|D\eta|^2 \\
& + \int_\Omega \left(1 + \frac{4}{\nu}\right)\frac{n^4N^2c_6^2}{(1+|u^\gamma|)^{2q}}\left[\phi'(|u^\gamma|)|u^\gamma| + \phi(|u^\gamma|)\right](1+|Du|)^2\eta^2.
\end{aligned}$$

Let us consider $p \in (0, +\infty)$ and let us assume that $|u^\gamma|^{2(p+1)} \in L^1(B_R)$, and for $t \in [0, +\infty)$ we set $\psi(t) = (p+1)^2 t^{2p}$. We approximate ψ as the same as in [1]. We consider $\psi_k(s) = \int_0^s \theta_k(t)dt$, when $p < 1/2$, we take

$$\theta_k(t) = \begin{cases} \psi'(\frac{1}{k}) & if\ t \in [0, \frac{1}{k}) \\ \psi'(t) & if\ t \in [\frac{1}{k}, k] \\ \psi'(k)(k+1-t) & if\ t \in (k, k+1) \\ 0 & if\ t \in [k+1, +\infty); \end{cases}$$

and when $p \geq 1/2$, we take

$$\theta_k(t) = \begin{cases} \psi'(t) & if\ t \in [0, k] \\ \psi'(k)(k+1-t) & if\ t \in (k, k+1) \\ 0 & if\ t \in [k+1, +\infty). \end{cases}$$

Now we can use $\phi = \psi_k$, we remark that $\psi_k(t) \leq \psi_k'(t)t + \psi_k(t) \leq (p+1)^2(2p+1)t^{2p}$ and we can obtain

$$\begin{aligned}
& \frac{\nu}{2}\int_\Omega \psi_k(|u^\gamma|)(Du^\gamma)^2\eta^2 dx \\
\leq\ & (\nu + nc_3)(p+1)^2(2p+1)\theta^{2p}\int_\Omega |Du^\gamma|^2\eta^2 dx \\
& + \left(1 + \frac{4n^2c_3^2}{\nu}\right)\int_\Omega (p+1)^2(2p+1)|u^\gamma|^{2(p+1)}|D\eta|^2 dx \\
& + 2\int_\Omega \left(1 + \frac{4}{\nu}\right)\frac{n^4N^2c_6^2}{(1+|u^\gamma|)^{2q}}(p+1)^2(2p+1)|u^\gamma|^{2p}(1+|Du|^2)\eta^2 dx.
\end{aligned} \tag{2.7}$$

We need that $p \leq q$ in order to have $\frac{|u^\gamma|^{2q}}{(1+|u^\gamma|)^{2q}} \leq 1$. So we get

$$\begin{aligned}
&\frac{\nu}{2}\int_\Omega |u^\gamma|^{2p}(Du^\gamma)^2\eta^2 dx \\
\leq\ & \left((\nu+nc_3)\theta^{2p} + \left(1+\frac{4}{\nu}\right)(2n^4N^2c_6^2)\right)(2p+1)||Du||^2_{L^2(\Omega)} \\
&+\left(1+\frac{4n^2c_3^2}{\nu}\right)(2p+1)\frac{4}{(R-\rho)^2}\int_\Omega |u^\gamma|^{2(p+1)}dx \\
&+2\left(1+\frac{4}{\nu}\right)(n^4N^2c_6^2)(2p+1)|\Omega|.
\end{aligned} \tag{2.8}$$

We set $\omega = |u^\gamma|^{p+1}\eta$, then $\omega \in W_0^{1,2}(B_R)$ and

$$|D\omega|^2 \leq 2(p+1)^2|u^\gamma|^{2p}|Du^\gamma|^2\eta^2 + 2n|u^\gamma|^{2(p+1)}\left(\frac{2}{R-\rho}\right)^2. \tag{2.9}$$

From (2.17) and (2.16), we get

$$\begin{aligned}
&\int_\Omega |D\omega|^2 dx \\
\leq\ & \frac{4}{\nu}(p+1)^2\left((\nu+nc_3)\theta^{2p} + \left(1+\frac{4}{\nu}\right)(n^4N^2c_6^2)\right)(2p+1)||Du||^2_{L^2(\Omega)} \\
&+\left(\frac{4}{\nu}(p+1)^2\left(1+\frac{4n^2c_3^2}{\nu}\right)(2p+1)+2n\right)\frac{4}{(R-\rho)^2}\int_\Omega |u^\gamma|^{2(p+1)}dx \\
&+\frac{4}{\nu}(p+1)^2\left(1+\frac{4}{\nu}\right)(2n^4N^2c_6^2)(2p+1)|\Omega|.
\end{aligned} \tag{2.10}$$

Now we use Sobolev embedding and the properties of η in order to get

$$\begin{aligned}
&\int_{B_\rho} |u^\gamma|^{(p+1)2^*}dx \leq \int_{B_R} ||u^\gamma|^{(p+1)}\eta|^{2^*}dx \\
=\ & \int_{B_R} |\omega|^{2^*}dx \leq \left[\frac{2(n-1)}{n-2}\int_{B_R}|D\omega|^2dx\right]^{\frac{2^*}{2}} dx \\
\leq\ & \left[\frac{4}{\nu}(p+1)^2\left((\nu+nc_3)\theta^{2p} + \left(1+\frac{4}{\nu}\right)(n^4N^2c_6^2)\right)(2p+1)||Du||^2_{L^2(\Omega)}\right. \\
&+\left(\frac{4}{\nu}(p+1)^2\left(1+\frac{4n^2c_3^2}{\nu}\right)(2p+1)+2n\right)\frac{4}{(R-\rho)^2}\int_\Omega |u^\gamma|^{2(p+1)} \\
&\left.+\frac{4}{\nu}(p+1)^2\left(1+\frac{4}{\nu}\right)(2n^4N^2c_6^2)(2p+1)|\Omega|\right]^{\frac{2^*}{2}} \times \left(\frac{2(n-1)}{n-2}\right)^{\frac{2^*}{2}}.
\end{aligned} \tag{2.11}$$

If for some $p \in (0,+\infty)$ with $p \leq q$ and for some $0 < \rho < R \leq 1$ with $\overline{B_R} \subset \Omega$, we have

$$|u^\gamma|^{2(p+1)} \in L^1(B_R), \tag{2.12}$$

then it results that

$$|u^\gamma|^{2^*(p+1)} \in L^1(B_\rho). \tag{2.13}$$

Since $u \in W^{1,2}(\Omega, R^N)$ and $\overline{B_R} \subset \Omega$, Sobolev embedding gives us

$$|u^\gamma|^{\frac{2n}{n-2}} \in L^1(B_R), \tag{2.14}$$

thus (2.12) is fulfilled with $p = min\{\frac{2}{n-2}, q\}$, this improves the integrability according to (2.13), the procedure can be iterated and following [2], after a finite numbers of steps, we reach the desired integrability. The ends the proof of Theorem 1.1.

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