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Iterative algorithms for totally quasi-ϕ-asymptotically nonexpansive mappings and monotone operators in Banach spaces

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Abstract

The purpose of this paper is to introduce a iterative sequence for finding a common element of the set of fixed points of a totally quasi-ϕ-asymptotically nonexpansive mapping and the set of zeros of an inverse-strongly monotone operator in a Banach space. We show the strong convergence of the given iterative sequence.

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1 Introduction

Let E be a real Banach space with dual space E^* and C be a closed and convex subset of E. For all $x \in E$ and $x^* \in E^*$, $\langle x, x^* \rangle$ denotes the generalized duality pairing. The normalized duality mapping $J : E \to E^*$ is defined by

$$J(x) = \{x^* \in E : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\}, \forall x \in E.$$

Let $A : C \to E^*$ be a nonlinear operator. The classical variational inequality for A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $VI(C,A)$. Such a problem is connected with the convex minimization problem, the complementarity, the problem of finding a point $x^* \in E$ satisfying $0 = Ax^*$.

The variational inequality (1) has been studied by many authors. If E is a Hilbert space, the metric projection operator $P_C : E \to E$ plays a very important role in solving the variational inequality (1). In general Banach spaces, the metric projection operator may not be defined. Alber [6] introduced that the generalized projection $\pi_C : E^* \to E$ and $\Pi_C : E \to E$ in uniformly convex and uniformly smooth spaces. In [23], by applying the general projection operator $\pi_C : E^* \to E$, J. L. studied the existence of the solution of the variational inequality

$$\langle Ax - \xi, y - x\rangle \geq 0, \forall y \in C.$$

By using the general projection operator $\Pi_C : E \to C$, Iiduka and Takahashi [11] introduced the iterative scheme for finding the solution of the variational inequality problem (1) for an inverse-strongly monotone operator A in a Banach space: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - Ax_n), \forall n \geq 1.$$

They proved that if J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to some point $z \in VI(C,A)$, where $z = \lim_{n\to\infty} \Pi_{VI(C,A)} x_n$.

The notion of monotone mapping was introduced by Zarantonello [3] , G.J. Minty [4] and Kacurovskii [5] in Hilbert space. This notion has been extended to Banach spaces by several authors (see [6, 7, 8, 9, 10]).

We recall that a mapping $A : E \to E^*$ is said to be

(1) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \forall x, y \in C.$$

(2) α-inverse-strongly monotone if

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C,$$

where $\alpha > 0$.

Take a functional $\phi : E \times E \to \Re$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,$$

for all $x, y \in E$. In a Hilbert space, $J = I$, where I is identity mapping, $\phi(x, y) = \|x - y\|^2$.

Let $T : C \to C$ be a mapping. A point $p \in C$ is called an asymptotic fixed point of T, if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

A mapping T is said to be

(1) relatively nonexpansive [17, 18], if $\widehat{F}(T) = F(T)$ and

$$\phi(p, Tx) \leq \phi(p, x), \forall x \in C, p \in F(T),$$

where $\widehat{F}(T)$ is the asymptotic fixed point set of T.
(2) relatively asymptotically nonexpansive [19], if $\widehat{F}(T) = F(T)$ and there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \forall x \in C, p \in F(T);$$

(3) quasi-ϕ-nonexpansive , if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \forall x \in C, p \in F(T);$$

(4) quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \emptyset$ and

$$\phi(p, T^n x) \leq k_n \phi(p, x), \forall x \in C, p \in F(T);$$

(5) totally quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0, \mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \Re^+ \to \Re^+$ with $\psi(0) = 0$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + \nu_n \psi(\phi(p, x)) + \mu_n, \forall x \in C, p \in F(T).$$

Remark 1.1 Every relatively nonexpansive mapping implies a relatively quasi-nonexpansive mapping, a quasi-ϕ-nonexpansive mapping implies a quasi-ϕ-asymptotically nonexpansive mapping and a quasi-ϕ -asymptotically nonexpansive mapping implies a totally quasi-ϕ-asymptotically nonexpansive mapping, but the converses are not true.

Alber [6] introduced that the generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is a solution of the minimization problem

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(x, y).$$

The problem of finding a common element of the set of the variational inequalities for monotone operators in the framework of Hilbert spaces and Banach spaces has been intensively studied by many authors, please see [8, 12, 6].

In 2006, Wu and Huang [13] introduced a new generalized f-projection operator in Banach spaces. Let $G : C \times E^* \to \Re \cup \{+\infty\}$ be a function defined by

$$G(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 + 2\rho f(x),$$

where $x \in C, x^* \in E^*$, ρ is a positive number and $f : C \to \Re \cup \{+\infty\}$ is proper, convex and lower semicontinuous. From the definition of $G(x, x^*)$, Wu and Huang [13] studied the following properties of $G(x, x^*)$:
(1) $G(x, x^*)$ is convex and continuous with respect to x^* when x is fixed;

(2) $G(x, x^*)$ is convex and lower semicontinuous with respect to y when x^* is fixed.

We say that $\pi_C^f : E^* \to 2^C$ is a generalized f-projection operator if

$$\pi_C^f x^* = \{u \in C : G(u, x^*) = \inf_{y \in C} G(y, x^*), x^* \in E^*\}.$$

Wu and Huang [13] studied the properties of π_C^f.
Let E be a reflexive Banach space with dual space E^*, and C be a nonempty closed convex subset of E. Then the following statements hold:
(1) $\pi_C^f x^*$ is a nonempty, closed and convex subset of C for all $x^* \in E^*$;
(2) if E is smooth, then for all $x^* \in E^*$, $x \in \pi_C^f x^*$ if and only if

$$\langle x - y, x^* - Jx\rangle + \rho f(y) - \rho f(x) \geq 0, \forall y \in C;$$

(3) if E is strictly convex and f is positive homogeneous (i.e., $f(tx) = tf(x)$) for all $t > 0$ such that $tx \in C$, then $\pi_C^f x^*$ is a single-valued mapping (this property is also can be seen in [20]).

It is well known that J is a single-valued mapping when E is a smooth space. There exists a unique element $x^* \in E^*$ such that $x^* = Jy$, $y \in E$. So, we can define the following function

$$H(x, y) = G(x, Jy) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 + 2\rho f(x).$$

We consider the second generalized f-projection operator in Banach spaces. $\Pi_C^f : E \to 2^C$ is a generalized f-projection oprator if

$$\Pi_C^f x = \{u \in C : G(u, Jx) = \inf_{y \in C} G(y, Jx), \forall x \in E\}.$$

If $f(y) > 0$ for all $y \in C$ and $f(0) = 0$, then the definition of totally quasi-ϕ-asymptotically nonexpansive mapping T is equivalent to the following:
If $F(T) \neq \emptyset$ and there exist nonnegative sequences ν_n, μ_n with $\nu_n \to 0, \mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \Re^+ \to \Re^+$ with $\varphi(0) = 0$ such that:

$$G(p, JT^n x) \leq G(p, Jx) + \nu_n \varphi(G(p, Jx)) + \mu_n, \forall x \in C, p \in F(T).$$

In 2013, S. Saewan et al. [16] introduced a new hybrid projection algorithm by the generalized f-projection operator for a countable family of totally quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property.

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JT_i^n x_n) \\ C_{n+1,i} = \{u \in C_n : G(u, Jy_{n,i}) \leq G(u, Jx_n) + \beta_n\} \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n,i}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_1. \end{cases}$$

They proved $\{x_n\}$ strongly converges to a point $\Pi^f_{\bigcap_{i=1}^{\infty} F(T_i)} x_1$ under suitable conditions.

Motivated by Zegeye and Shahzad [14], Wu and Huang [13], and S. Saewan et al. [16], we introduce a new scheme for finding the common element of the zero of a inverse strongly monotone operator and the fixed point set of a totally quasi-ϕ-asymptotically nonexpansive mapping, and prove the strong convergence of the scheme under suitable conditions.

2 Preliminary Notes

Let E be a real Banach space. The modules of smoothness of E is defined by the function $\rho_E(\tau) : [0, +\infty) \to [0, +\infty)$,

$$\rho_E(\tau) := \sup\{\frac{\|x - y\| + \|x + y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\}.$$

If $\rho_E(\tau) > 0, \forall \tau > 0$, E is called smooth, and E is said to be uniformly smooth if and only if

$$\lim_{t \to 0^+} \frac{\rho_E(t)}{t} = 0.$$

Let $B = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. E is said to be strictly convex if for any $x, y \in B$, $x \neq y$, implies $\|\frac{x+y}{2}\| < 1$. It is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in B$, $\|x - y\| \geq \varepsilon$ implies $\|\frac{x+y}{2}\| < 1 - \delta$.

It is well known that a uniformly convex Banach space is reflexive and strictly convex. The modules of convexity of E is a function $\delta_E : (0, 2] \to [0, 1]$:

$$\delta_E(\varepsilon) := \inf\{1 - \frac{\|x - y\| + \|x + y\|}{2}\| : x, y \in B, \|x - y\| = \varepsilon\}.$$

We can know that E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let $p > 1$, E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) > c\varepsilon^p$ for all $\varepsilon \in (0, 2]$. Every p-uniformly convex Banach space is a uniformly convex Banach space.

Some basic properties of E, E^*, J and J^{-1} are as follows (see [1, 2]):
(1) if E is a uniformly smooth Banach space, then J is uniformly norm-to-norm continuous on each bounded set of E;
(2) if E is a reflexive, smooth and strictly convex Banach space, then the normalized duality mapping J is single-valued, one-to-one and onto;
(3) if E is a reflexive, smooth and strictly convex Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-valued, bijective and is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$, $J^{-1}J = I_E$.
(4) if E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous.

Recall that a Banach space E has Kadec-Klee property: if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

Definition 2.1 *if E is a uniformly smooth Banach space, then J is uniformly norm-to-norm continuous on each bounded set of E;*

Definition 2.2 *if E is a reflexive, smooth and strictly convex Banach space, then the normalized duality mapping J is single-valued, one-to-one and onto;*

Definition 2.3 *if E is a reflexive, smooth and strictly convex Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-valued, bijective and is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$, $J^{-1}J = I_E$.*

Definition 2.4 *if E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous.*

Recall that a Banach space E has Kadec-Klee property: if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.
if E is a reflexive, smooth and strictly convex Banach space, then the normalized duality mapping J is single-valued, one-to-one and onto;

3 Main Results

In the sequel, we need the following results.

Lemma 3.1 *[22]Let E be a 2-uniformly convex Banach space. Then, for all $x, y \in E$, we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|, \tag{2}$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

Lemma 3.2 *[6]Let E be a real reflexive, strictly convex and smooth Banach space, C be a nonempty closed and convex subset of E. Let $x \in E$, then $\forall y \in C$,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x). \tag{3}$$

Lemma 3.3 *[9]Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Applying the definition of ϕ and J, we define the functional $V : E \times E^* \to \Re$ studied in[6] by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2$$

for all $x \in E$ and $x^* \in E^*$, that is $V(x, x^*) = \phi(x, J^{-1}x^*)$. We know the following result:

Lemma 3.4 *Let E be a real reflexive, strictly convex, smooth Banach space with dual space E^*, then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le V(x, x^* + y^*), \tag{4}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

In [15], Li et al. introduced the following properties of Π_C^f:

Lemma 3.5 *Let E be a reflexive Banach smooth space and C be a nonempty, closed and convex subset of E. The following statements hold:*
(1) Π_C^f is nonempty, closed and convex subset of C for all $x \in E$;
(2) for all $x \in E$, $\widehat{x} \in \Pi_C^f x$ if and only if

$$\langle \widehat{x} - y, Jx - J\widehat{x}\rangle + \rho f(y) - f(x) \ge 0, \forall y \in C;$$

(3) if E is strictly convex, then Π_C^f is a single-valued mapping.

Lemma 3.6 *[15] Let E be a real reflexive smooth Banach space and C be a nonempty closed and convex subset of E. If $\widehat{x} \in \Pi_C^f x$ for all $x \in E$, then*

$$\phi(y, \widehat{x}) + G(\widehat{x}, Jx) \le G(y, Jx), \forall y \in C.$$

We also need the following Lemmas for the proof of our main results.

Lemma 3.7 *[21] Let E be a Banach space and $f : E \to \Re \cup \{+\infty\}$ be a lower semicontinuous convex function. Then there exists $x^* \in E^*$ and $\beta \in \Re$ such that*

$$f(x) \ge \langle x, x^*\rangle + \beta, \forall x \in E.$$

Lemma 3.8 *[24] Let C be a nonempty closed convex subset of a uniformly smooth and strictly convex Banach space E with Kadec-Klee property. Let $T : C \to C$ be a closed and totally quasi-ϕ-asymptotically nonexpansive mapping with μ_n and ν_n of nonnegative real numbers with $\mu_n \to 0, \nu_n \to 0$ and a strictly increasing continuous function $\psi : \Re^+ \to \Re^+$ with $\psi(0) = 0$. If $\mu_1 = 0$, then the fixed points set $F(T)$ is a closed convex subset of C.*

Lemma 3.9 *[25] Let E be a real smooth Banach space, $A : E \to 2^{E^*}$ be a maximal monotone mapping, then $A^{-1}(0)$ is a closed and convex subset of E and the graph of A, $G(A)$, is demiclosed in the following sense: $\forall \{x_n\} \subset D(A)$ with $x_n \rightharpoonup x$ in E, and $\forall y_n \in Ax_n$ with $y_n \to y$ in E^* implies that $x \in D(A)$ and $y \in Ax$.*

In order to prove our results, we make use the following function $H^*(x, x^*) : E \times E^* \to \Re$ defined by

$$H^*(x, x^*) = G(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 + 2\rho f(x), \forall x \in E, x^* \in E^*.$$

That is $H^*(x, x^*) = H(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. Using the definition of $H^*(x, x^*)$,Using the definition of $H^*(x, x^*)$, in according to Lemma 3.4, we can have:

Lemma 3.10 *Let E be a reflexive strictly convex and smooth Banach space with its dual space E^*, then*

$$H^*(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le H^*(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Theorem 3.11 *Let E be a real uniformly smooth and 2-uniformly convex Banach space with dual space E^* and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an inverse strongly monotone operator with constant γ, and $T_i : C \to C, (i = 1, 2, \cdots)$ be a countable family of closed and uniformly totally quasi-ϕ-asymptotically nonexpansive mapping with the sequence ν_n and μ_n of nonnegative real numbers and $\nu_n \to 0, \mu_n \to 0$ as $n \to \infty$. A strictly increasing continuous function $\psi : \Re^+ \to \Re^+$ with $\psi(0) = 0$, and assume that T_i is uniformly asymptotically regular for all $i \ge 1$ with $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and such that $\Sigma = A^{-1}(0) \cap \mathcal{F} \ne \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_C^f x_0$ and $C_{1,i} = C$ and $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^{-1}(Jx_n - \alpha_n Ax_n) \\ z_{n,i} = T_i^n y_n \\ C_{n+1,i} = \{u \in C_n : G(u, Jz_{n,i}) \le G(u, Jx_n) + \beta_n\} \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i} \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0. \end{cases}$$

where $0 < \alpha_n \le b_0 := \frac{\gamma c^2}{2}$ and $\beta_n = \nu_n \sup \psi(G(p, Jx_n)) + \mu_n$, J is the normalized duality mapping on E, then the sequence $\{x_n\}$ is well defined for each $n \ge 1$ and converges strongly to $\Pi_\Sigma^f x_0$.

Proof: We first show that C_{n+1} is closed and convex. From the definition, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i} = C$ is closed and convex. Suppose that $C_{n,i}$ is closed and convex, for any $z \in C_{n,i}$, note that

$$H(u, z_{n,i}) \leq H(u, x_n) + \beta_n$$

is equivalent to

$$2\langle z, Jx_n - Jz_{n,i}\rangle \leq \|x_n\|^2 - \|z_{n,i}\|^2 + \beta_n, \forall i \geq 1.$$

So, $C_{n+1,i}$ is closed and convex, hence $C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}$ is closed and convex for all $n \geq 1$.

Next we show that $\Sigma = A^{-1}(0) \cap \mathcal{F} \subset C_n$. Let $p \in \Sigma$, by T_i is a totally quasi-ϕ-asymptotically nonexpansive mapping, A is a γ-inverse-strongly monotone operator, we have

$$\begin{aligned}
& G(p, Jy_n) = H(p, y_n) \\
= \; & H(p, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
= \; & H^*(p, Jx_n - \lambda_n Ax_n) \\
\leq \; & H^*(p, Jx_n - \lambda_n Ax_n + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - p, \lambda_n Ax_n\rangle \\
= \; & H^*(p, Jx_n) - 2\langle y_n - p, \lambda_n Ax_n\rangle \\
= \; & H(p, x_n) - 2\lambda_n\langle x_n - p, \lambda_n(Ax_n - Ap)\rangle - 2\lambda_n\langle y_n - x_n, Ax_n - Ap\rangle \\
\leq \; & H(p, x_n) - 2\lambda_n\langle x_n - p, Ax_n - Ap\rangle + 2\lambda_n\|y_n - x_n\|\|Ax_n - Ap\| \\
\leq \; & H(p, x_n) - 2\lambda_n\gamma\|Ax_n - Ap\|^2 + 2\lambda_n\|J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}Jx_n\| \\
& \times\|Ax_n - Ap\| \\
\leq \; & H(p, x_n) - 2\lambda_n\gamma\|Ax_n - Ap\|^2 + 2\lambda_n\|JJ^{-1}(Jx_n - \lambda_n Ax_n) - JJ^{-1}Jx_n\| \\
& \times\|Ax_n - Ap\| \\
\leq \; & H(p, x_n) - 2\lambda_n\gamma\|Ax_n - Ap\|^2 + \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2 \\
\leq \; & H(p, x_n) - 2\lambda_n(\gamma - \frac{2}{c^2}\lambda_n)\|Ax_n - Ap\|^2 \\
\leq \; & H(p, x_n) = G(p, Jx_n). \qquad (5)
\end{aligned}$$

By the definition of totally quasi-ϕ-asymptotically nonexpansive mapping and the property of ψ, we obtain

$$\begin{aligned}
G(p, Jz_{n,i}) = G(p, JT_i^n y_n) & \leq G(p, Jy_n) + \nu_n\psi(G(p, Jy_n)) + \mu_n \\
& \leq G(p, Jx_n) + \nu_n\psi(G(p, Jx_n)) + \mu_n \\
& = G(p, Jx_n) + \beta_n \qquad (6)
\end{aligned}$$

this shows that $p \in C_{n+1}$, which implies $\Sigma := \mathcal{F} \cap A^{-1}(0) \subset C_{n+1}$, hence $\Sigma \subset C_n$.

step 1. *$\{x_n\}$ is bounded sequence. Since $f : E \to \Re$ is convex and lower semicontinuous function, from Lemma 3.7, there exists $x^* \in E^*$ and $\beta \in \Re$ such that $f(x) \geq \langle x, x^*\rangle + \beta, \forall x \in E$. From $x_n \in E$, it follows that*

$$\begin{aligned} G(x_n, Jx_0) &= \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_n) \\ &\geq \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho\langle x_n, x^*\rangle + 2\rho\beta \\ &\geq \|x_n\|^2 - 2\langle x_n, Jx_0 - \rho x^*\rangle + \|x_0\|^2 + 2\rho\beta \\ &\geq \|x_n\|^2 - 2\|x_n\|\|Jx_0 - \rho x^*\| + \|x_0\|^2 + 2\rho\beta \\ &= (\|x_n\| - \|Jx_0 - \rho x^*\|)^2 + \|x_0\|^2 + 2\rho\beta - \|Jx_0 - \rho x^*\|^2. \end{aligned}$$

For any $p \in \Sigma$, from $x_n = \Pi^f_{C_n} x_0$, we have

$$(\|x_n\| - \|Jx_0 - \rho x^*\|)^2 + \|x_0\|^2 + 2\rho\beta - \|Jx_0 - \rho x^*\|^2 \leq G(x_n, Jx_0) \leq G(p, Jx_0)$$

i.e.,

$$\begin{aligned} (\|x_n\| - \|Jx_0 - \rho x^*\|)^2 &\leq G(p, Jx_0) - \|x_0\|^2 - 2\rho\beta + \|Jx_0 - \rho x^*\|^2 \\ &\leq G(p, Jx_0) + \|Jx_0 - \rho x^*\|^2. \end{aligned}$$

This implies that$\{x_n\}$ is bounded and so are $\{y_n\}, \{z^i_n\}$.

Step 2. *$\{x_n\}$ strongly converges to a point $q \in C$.*

Since $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, and $x_n = \Pi^f_{C_n} x_0$, from lemma 3.6, we have

$$0 \leq \|x_{n+1} - x_n\|^2 \leq \phi(x_{n+1}, x_n) \leq G(x_{n+1}, Jx_0) - G(x_n, Jx_0).$$

So, the $\{G(x_n, Jx_0)\}$ is nondecreasing and bounded, this implies that $\lim_{n\to\infty} G(x_n, Jx_0)$ exists. We also obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{7}$$

Since $\{x_n\}$ is bounded and E is reflexive Banach space, C_n is bounded and convex subset of E, we can have $x_n \rightharpoonup q \in C_n$. Next we show that $x_n \to q$. From $x_n = \Pi^f_{C_n} x_0$ and $q \in C_n$, we get

$$G(x_n, Jx_0) \leq G(q, Jx_0),$$

and from f is convex and lowercontinuous, we have

$$\begin{aligned} \liminf_{n\to\infty} G(x_n, Jx_0) &= \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_n) \\ &\geq \|q\|^2 - 2\langle q, Jx_0\rangle + \|x_0\|^2 + 2\rho f(q) \\ &= G(q, Jx_0). \end{aligned}$$

So,

$$G(q, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(q, Jx_0).$$

That is $\lim_{n\to\infty} G(x_n, Jx_0) = G(q, Jx_0)$, *which implies that* $\|x_n\| \to \|q\|$, *by the virtue of the Kadec-Klee property of* E, *it follows that*

$$\lim_{n\to\infty} x_n = q.$$

Step 3. *We show that* $q \in \Sigma$.
Since $\{x_n\}$ *is bounded, it follows that* $\lim_{n\to\infty} \beta_n = 0$.
From $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, *we have*

$$\begin{aligned} G(x_{n+1}, Jz_{n,i}) &\leq G(x_{n+1}, Jx_n) + \beta_n \\ &\Leftrightarrow \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jz_{n,i}\rangle + \|z_{n,i}\| + 2\rho f(x_{n+1}) \\ &\quad \leq \|x_{n+1}\|^2 - 2\langle x_n, Jx_n\rangle + \|x_n\| + 2\rho f(x_{n+1}) + \beta_n \\ &\Leftrightarrow \phi(x_{n+1}, z_{n,i}) \leq \phi(x_{n+1}, x_n) + \beta_n \end{aligned}$$

So, from (7) and $\lim_{n\to\infty} \beta_n = 0$, *we have* $\phi(x_n, z_{n,i}) \to 0$, *i.e.,*

$$\|x_{n+1} - z_{n,i}\| \to 0.$$

We also obtain

$$\|x_n - z_{n,i}\| \leq \|x_{n+1} - x_n\| + \|x_n - z_{n,i}\| \to 0. \tag{8}$$

Take $p \in \Sigma$, *from (5), we have*

$$\begin{aligned} & H(p, z_{n,i}) = G(p, T_i^n y_n) \\ \leq\ & H(p, y_n) + \nu_n \psi(H(p, y_n)) + \mu_n \\ \leq\ & H(p, x_n) - 2\lambda_n(\gamma - \frac{2}{c^2}\lambda_n)\|Ax_n - Ap\|^2 + \nu_n \psi(H(p, y_n)) + \mu_n, \end{aligned} \tag{9}$$

i.e., $0 < a < \lambda_n < b = \frac{c^2\gamma}{2}$,

$$2\lambda_n(\gamma - \frac{2}{c^2}\lambda_n)\|Ax_n - Ap\|^2 \leq H(p, x_n) - H(p, z_{n,i}) + \nu_n \psi(H(p, y_n)) + \mu_n,$$

further,

$$\begin{aligned} & 2a(\gamma - \frac{2}{c^2}b)\|Ax_n - Ap\|^2 \\ \leq\ & H(p, x_n) - H(p, z_{n,i}) + \nu_n \psi(H(p, y_n)) + \mu_n \\ =\ & 2\langle p, Jz_{n,i} - Jx_n\rangle + (\|x_n\|^2 - \|z_{n,i}\|^2) + \nu_n \psi(H(p, y_n)) + \mu_n, \end{aligned} \tag{10}$$

notice that $\nu_n \to 0, \mu_n \to 0$, $\|x_n - z_{n,i}\| \to 0$ *and* $\{H(q, y_n)\}$ *is bounded, we have* $\|Ax_n - Ap\| \to 0$.

From Lemma 3.1, we obtain

$$\begin{aligned} 2\langle y_n - x_n, -\lambda_n A x_n\rangle &= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}Jx_n, -\lambda_n Ax_n\rangle \\ &\leq 2\|J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}Jx_n\|\|\lambda_n Ax_n\| \\ &\leq \frac{4}{c^2}\|JJ^{-1}(Jx_n - \lambda_n Ax_n) - JJ^{-1}Jx_n\|\|\lambda_n Ax_n\| \\ &= \frac{4}{c^2}\lambda_n^2\|Ax_n\|^2 \leq \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2. \qquad (11) \end{aligned}$$

Applying Lemma 3.4, (11), we have

$$\begin{aligned} \phi(x_n, y_n) &= \phi(x_n, J^{-1}(Jx_n - \lambda_n Ax_n)) \\ &= V(x_n, Jx_n - \lambda_n Ax_n) \\ &\leq V(x_n, Jx_n - \lambda_n Ax_n + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\rangle \\ &= \phi(x_n, x_n) + 2\langle y_n - x_n, -\lambda_n Ax_n\rangle \\ &\leq \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2 \end{aligned}$$

Therefore, $\phi(x_n, y_n) \to 0$, *i.e.,* $\|x_n - y_n\| \to 0$.

$$\|T_i^n y_n - q\| = \|z_{n,i} - q\| \leq \|z_{n,i} - x_n\| + \|x_n - q\| \to 0.$$

$$\begin{aligned} \|T_i^n x_n - q\| &\leq \|T_i^n x_n - T_i^n y_n\| + \|T_i^n y_n - q\| \\ &\leq L\|x_n - y_n\| + \|T_i^n y_n - q\| \to 0. \qquad (12) \end{aligned}$$

From the uniformly asymptotically regular of T, *we have*

$$\|T_i^{n+1}x_n - q\| \leq \|T_i^{n+1}x_n - T^n x_n\| + \|T_i^n x_n - q\| \to 0. \qquad (13)$$

i.e., $T(T^n x_n) \to p$. *From the continuity and closedness of* T, *we have* $Tp = p$, *so,* $p \in F(T)$. *Since the normalized duality mapping* J *is uniformly continuous on bounded set, we have* $\lim_{n\to\infty} \lambda_n\|Ax_n\| = \lim_{n\to\infty} \|Jy_n - Jx_n\| \to 0, (n \to \infty)$, *then* $Ax_n \to 0$. *Since* A *is Lipschitz continuous and monotone, it is maximal monotone (see, e,g., [1]), so by the Lemma 3.9, we have* $q \in A^{-1}0$.

Step 4. *we show that* $q = \Pi_F^f x_0$.

Since F *is closed and convex,* $\Pi_F^f x_0$ *is single-valued, which is denoted by* w. *By the definition* $x_n = \Pi_{C_n}^f x_0$ *and* $w \in F \subset C_n$, *we obtain*

$$G(x_n, Jx_0) \leq G(w, Jx_0).$$

By the definition of G *and* f, *we can know that* $G(\xi, Jx)$ *is convex and lower semicontinuous with respect to* ξ *and so*

$$G(q, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(w, Jx_0).$$

From the definition of $\Pi_F^f x_0, q \in F$, *we have* $w = q = \Pi_F^f x_0$.

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