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The characterization of regular ordered semigroups by $(\in, \in \vee q)-$fuzzy ideals

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Abstract

In this paper, based on the inclusion and quasi-coincident relation of fuzzy sets, we introduce and investigate the concepts of $(\in, \in \vee q)-$*fuzzy quasi-ideals* and *interior ideals*. Also, the characterization of regular ordered semigroups in terms of $(\in, \in \vee q)-$*fuzzy left ideals* (*right ideals, bi-ideals, quasi-ideals, interior ideals*) is also studied.

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1 Introduction

The concept of fuzzy set was introduced by Zadeh[1] in 1965.Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic,set theory, groups theory, groupoids, real analysis, measure theory, topology, ect. Ordered semigroups have many applications in the theory of sequential machines, formal languages, computer arithmetics, and error-correcting codes. Based on the terminology given by Zadeh, fuzzy sets in ordered groupiods and semigroups have been first considered by Kehayopulu and Tsingsgelis [2-9].Using the notion "belongingness$(\in)$" and "quasi-coincidence (q)" of a fuzzy point with a fuzzy set introduced by Pu and Liu [10].The detailed study with $(\in, \in \vee q)-$fuzzy subgroup has been considered in Bhakat and Das[11].In particular, $(\in, \in \vee q)-$fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. Jun and Khan [12-14] gave the concept of a generalized fuzzy bi-ideal in ordered semigroups and characterized regular ordered semigroups in terms of this notion. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic constructions of the existing fuzzy subsystems of other algebraic structures (see[15-18]).

As a further study, we will study the characterization of regular ordered semigroups. Based on the inclusion and quasi-coincident relation of fuzzy sets, this paper introduces the concepts of

$(\in, \in \vee q)$ – fuzzy quasi-ideals and interior ideals. We also give some characterizations of regular ordered semigroups in terms of $(\in, \in \vee q)$ – fuzzy left ideals (right ideals, bi-ideals, quasi-ideals interior ideals).

2 Preliminaries

In this section, we will recall some basic notions, results on ordered semigroups and fuzzy sets.

2.1 Ordered semigroup

An ordered semigroup is an algebraic system $(S, ., \leq)$ consisting of a non-empty set S together with a binary operation "+" and a compatible ordering "$\leq$" on S such that $(S, .)$ is a semigroup and $x \leq y$ implies $ax \leq ay$ and $xa \leq ya$ for all $x, y, a \in S$ .Let $(S, ., \leq)$ be an ordered semigroup. A subset A of S is called a *left* (resp. *right*) *ideal* of S if it satisfies the following conditions:

(1) $SA \subseteq A$ (resp., $AS \subseteq A$);

(2) if $x \in A$ and $y \in A, y \leq x$, then $y \in A$.[4]

If A is both a left and a right ideal of S, then A is called an *ideal* of S.

Let $(S, ., \leq)$ be an ordered semigroup. A subset P of S is called a *bi-ideal* if it satisfies the following conditions:

(1) $PP \subseteq P$;

(2) $PSP \subseteq P$;

(3) if $x \in P$ and $S \ni y \leq x$, then $y \in P$. [4]

Let $(S, ., \leq)$ be an ordered semigroup. A subset Q of S is called a *quasi-ideal* if it satisfies the following conditions

(1) $QS \cap SQ \subseteq Q$;

(2) if $x \in Q$ and $S \ni y \leq x$, then $y \in Q$. [2]

Let $(S,.,\leq)$ be an ordered semigroup. A subset B of S is called an *interior ideal* if it satisfies the following conditions:

(1) $BB \subseteq B$;

(2) $SBS \subseteq B$;

(3) if $x \in B$ and $S \ni y \leq x$, then $y \in B$. [3]

2.2 Fuzzy sets

Let X be a non-empty set and A a subset in X. The characteristic function of A is the function χ_A of X into $[0,1]$ defined by $\chi_A(x)=1$ if $x \in A$ and $\chi_A(x)=0$ otherwise.

A fuzzy set μ of a non-empty set X is defined as a mapping from X into $[0,1]$, where $[0,1]$ is the usual interval of real numbers. The set of all fuzzy subsets of X is denoted by $F(X)$. For any $A \subseteq X$ and $r \in (0,1]$, the fuzzy subset μ of X defined by

$$\mu(y)=\begin{cases} r, y=x \\ 0, y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r.

For a fuzzy point x_r and a fuzzy subset μ of X, we say that:

(1) $x_r \in \mu$, if $\mu(x) \geq r$.

(2) $x_r q \mu$, if $\mu(x)+r>1$.

(3) $x_r \in q\mu$, if $x_r \in \mu$ or $x_r q \mu$.

Let us now introduce a new ordered relation on $F(X)$, called inclusion and quasi-coincident relation and denoted as "$\subseteq \vee q$", as follows.

For any $\mu, \nu \in F(X)$, by $\mu \subseteq \vee q \nu$ we mean that $x_r \in \mu$ implies $x_r \in \vee q \nu$ for all $x \in S$ and $r \in (0,1]$.

In what follows, unless otherwise stated, $\overline{\in \vee q}$ means $\in \vee q$ does not hold $\overline{\subseteq \vee q}$ implies $\subseteq \vee q$ is not true.

Lemma2.1 [19] Let $\mu, \nu \in F(X)$. Then $\mu \subseteq \vee q \nu$ if and only if $\nu(x) \geq \min\{\mu(x), 0.5\}$ for all $x \in S$.

Definition 2.1 [2] Let $(S, ., \leq)$ be an ordered semigroup and $\mu, \nu \in F(S)$. Define the product of μ and ν, denoted by $\mu \circ \nu$, as

$$(\mu \circ \nu)(x) = \begin{cases} \sup\limits_{x \leq yz} \min\{\mu(y), \nu(z)\} & \text{if} \quad x \leq yz \text{ for some } y, z \in S \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in S$.

Lemma2.3. [19] Let $(S, ., \leq)$ be an ordered semigroup and $A, B \subseteq S$ then

(1) $A \subseteq B$ if and only if $\chi_A \subseteq \vee q \chi_B$;

(2) $\chi_A \cap \chi_B = \chi_{A \cap B}$;

(3) $\chi_A \circ \chi_B = \chi_{(AB]}$.

3 $(\in, \in \vee q)-$ fuzzy ideals of an ordered semigroup

Definition 3.1 [19] Let $(S, ., \leq)$ be an ordered semigroup. A fuzzy subset μ of S is an $(\in, \in \vee q)-$*fuzzy left (resp., right)* ideal of S if it satisfies:

(F1a) $\mu \circ \chi_S \subseteq \vee q \mu$ $(resp., \chi_S \circ \mu \subseteq \vee q \mu)$;

(F2a) if $y \leq x$ and $x_t \in \mu \Rightarrow y_t \in \vee q \mu$ for all $x, y \in S$ and $t \in (0,1]$.

A fuzzy subset in S is called an $(\in, \in \vee q)-$*fuzzy ideal* of S if it is both an $(\in, \in \vee q)-$ *fuzzy right ideal* and an $(\in, \in \vee q)-$*fuzzy left ideal* of S.

Definition 3.2[19] Let $(S, ., \leq)$ be an ordered semigroup. A fuzzy subset μ of S is an $(\in, \in \vee q)-$*fuzzy bi-ideal* of S it satisfies conditions (F2a) and

(F3a) $\mu \circ \mu \subseteq \vee q \mu$;

(F4a) $\mu \circ \chi_S \circ \mu \subseteq \vee q \mu$.

Definition 3.3 Let $(S,.,\leq)$ be an ordered semigroup. A fuzzy subset μ of S is an $(\in,\in\vee q)-$*fuzzy quasi-ideal* of S it satisfies conditions (F2a) and

(F6a) $\mu\circ\chi_S\cap\chi_S\circ\mu\subseteq\vee q\mu$.

Definition 3.4 Let $(S,.,\leq)$ be an ordered semigroup. A fuzzy subset μ. of S is an $(\in,\in\vee q)-$*fuzzy interior ideal* of S it satisfies conditions (F2a) and

(F5a) $\chi_S\circ\mu\circ\chi_S\subseteq\vee q\mu$.

Lemma 3.1 [19] A fuzzy subset μ in an ordered semigroup S is an $(\in,\in\vee q)-$*fuzzy left (resp., right) ideal* of S if and only if it satisfies the following condition:

(F1b) $(\forall x,y\in S)\quad(\mu(xy)\geq\min\{\mu(x),0.5\}(resp.\mu(xy)\geq\min\{\mu(y),0.5\}))$;

(F2b) $(\forall x,y\in S)\ (y\leq x\Rightarrow\mu(y)\geq\min\{\mu(x),0.5\})$.

Lemma 3.2 [19] A fuzzy subset μ in an ordered semigroup S is an $(\in,\in\vee q)-$*fuzzy bi-ideal* of S if and only if it satisfies (F2b) and

(F3b) $(\forall x,y\in S)(\mu(xy)\geq\min\{\mu(x),\mu(y),0.5\})$;

(F4b) $(\forall x,y,z\in S)\ (\mu(xyz)\geq\min\{\mu(x),\mu(z),0.5\}$.

Lemma 3.3 A fuzzy subset μ in an ordered semigroup S is an $(\in,\in\vee q)-$*fuzzy interior ideal* of S if and only if it satisfies (F2b) and

(F5b) $(\forall x,y,z\in S)\quad(\mu(xyz)\geq\min\{\mu(y),0.5\}$.

Proof. The proof is similar to that of Lemma 3.2.

Theorem 3.1 Let S be an ordered semigroups and $\mu\in F(S$ .Then

(1) μ is an $(\in,\in\vee q)-$*fuzzy quasi-ideals (resp., interior ideals)* of S if and only if non-empty subset μ_r is a *quasi-ideals (resp., interior ideals)* of S for all $r\in(0,0.5]$;

(2) μ is an $(\in, \in \vee q)-$*fuzzy quasi-ideals (resp., interior ideals)* of S if and only if non-empty subset $<\mu>_r$ is a *quasi-ideals (resp., interior ideals)* of S for all $r \in (0.5,1]$;

(3) μ is an $(\in, \in \vee q)-$fuzzy *quasi-ideals (resp., interior ideals)* of S if and only if non-empty subset $[\mu]_r$ is a *quasi-ideals (resp., interior ideals*) of S for all $r \in (0,1]$.

Proof. We only show (3). (1) and (2) can be similarly proved. Let μ be an $(\in, \in \vee q)-$*fuzzy interior ideal* of S and assume that $[\mu]_r \neq \varnothing$ for some $r \in (0,1]$. Let $x, y, z \in [\mu]_r$. Then $x_r \in \vee q\mu$, $y_r \in \vee q\mu$, and $z_r \in \vee q\mu$ that is, $\mu(x) \geq r$ or $\mu(x) + r > 1$, $\mu(y) \geq r$ or $\mu(y) + r > 1$ and $\mu(z) \geq r$ or $\mu(z) + r > 1$. Since μ is an $(\in, \in \vee q)-$ *fuzzy interior ideal* of S, we have $\mu(xyz) \geq \min\{\mu(y), 0.5\}$. We consider the following cases:

Case 1 $r \in (0, 0.5]$. Then $1 - r \geq 0.5 \geq r$.

① If $\mu(x) \geq r$ or $\mu(y) \geq r$, or $\mu(z) \geq r$, then

$$\mu(xyz) \geq \min\{\mu(y), 0.5\} \geq r.$$

Hence $(xyz)_r \in \mu$.

② If $\mu(x) + r > 1$, $\mu(y) + r > 1$ and $\mu(z) + r > 1$, then

$$\mu(xyz) \geq \min\{\mu(y), 0.5\} = 0.5 \geq r.$$

Hence $(xyz)_r \in \mu$.

Case 2 $r \in (0.5,1]$. Then $r > 0.5 > 1 - r$.

If $\mu(x) + r > 1$, $\mu(y) + r > 1$ and $\mu(z) + r > 1$, then

$$\mu(xyz) \geq \min\{\mu(y), 0.5\} = 0.5 > 1 - r.$$

Hence $(xyz)_r q\mu$.

If $\mu(x) \geq r$ or $\mu(y) \geq r$, or $\mu(z) \geq r$, then

$$\mu(xyz) \geq \min\{\mu(y), 0.5\} > 1 - r.$$

Hence $(xyz)_r q\mu.$

Thus, in any case, $(xyz)_r \in \vee q\mu$, that is, $xyz \in [\mu]_r$. Therefore, $[\mu]_r$ is a *interior ideal* of S.

Conversely, assume that the given conditions hold. Let $x, z \in S$. If there exist $y \in S$ such that $\mu(xyz) < r = \min\{\mu(y), 0.5\}$, then $x_r \in \mu$ but $(xyz)_r \overline{\in \vee q\mu}$, that is, $x, y, z \in [\mu]_r$ but $(xyz) \notin [\mu]_r$, a condiction. Hence $\mu(xyz) \geq \min\{\mu(y), 0.5\}$ for all $x, y, z \in S$. Therefore μ is an $(\in, \in \vee q)-$*fuzzy interior ideal* of S.

Theorem 3.2 Let $(S,.,\leq)$ be an ordered semigroup. And $A \subseteq S$ Then A is *quasi-ideals (resp., interior ideals)* of S if and only if the fuzzy subset μ in S such that $\mu(x) \geq 0.5$ for all $x \in A$ and $\mu(x) = 0$ otherwise is an $(\in, \in \vee q)-$*fuzzy quasi-ideal (resp., interior ideal)* of S.

Proof. It is straightforward by Theorem 3.1

Lemma 3.4 [4,5] Let $(S,.,\leq)$ be an ordered semigroup and $A \subseteq S$. Then the following conditions hold:

(1) A is a *left (resp. right) ideal* of S if and only if χ_A is an $(\in, \in \vee q)-$*fuzzy left (resp. right) ideal* of S;

(2) A is a *bi-ideal* of S if and only if χ_A is an $(\in, \in \vee q)-$*fuzzy* $bi-ideal$ of S;

(3) A is a *quasi-ideal* of S if and only if χ_A is an $(\in, \in \vee q)-$*fuzzy quasi-ideal* of S;

(4) A is an *interior ideal* of S if and only if χ_A is an $(\in, \in \vee q)-$*fuzzy interior ideal* of S.

Proof It is straightforward.

Lemma 3.5 Let $(S,.,\leq)$ be an ordered semigroup. Then

(1) Every $(\in, \in \vee q)-$*fuzzy left (right) ideal* of S is an $(\in, \in \vee q)-$*fuzzy quasi-ideal* of S;

(2) Every $(\in, \in \vee q)-$*fuzzy quasi-ideal* of S is an $(\in, \in \vee q)-$*fuzzy bi-ideal* of S.

Proof The proof of (1) is straightforward. We show (2). Let μ be any $(\in, \in \vee q)-$*fuzzy quasi-ideals* of S. To show that μ is an $(\in, \in \vee q)-$*fuzzy bi-ideal* of S it is sufficient to show $\mu \circ \mu \subseteq \vee q \mu$ and $\mu \circ \chi_S \circ \mu \subseteq \vee q \mu$. In fact, since μ is an $(\in, \in \vee q)-$ *fuzzy quasi-ideal* of S, we have $\mu \circ \mu \subseteq \vee q(\chi_S \circ \mu)$, $\mu \circ \mu \subseteq \vee q \mu \circ \chi_S$, $\mu \circ \chi_S \circ \mu \subseteq \vee q \chi_S \circ \mu$ and $\mu \circ \chi_S \circ \mu \subseteq \vee q \mu \circ \chi_S$. Hence $\mu \circ \mu \subseteq \vee q \chi_S \circ \mu \bigcap \mu \circ \chi_S \subseteq \vee q \mu$ and $\mu \circ \chi_S \circ \mu \subseteq \vee q \chi_S \circ \mu \bigcap \mu \circ \chi_S \subseteq \vee q \mu$ .This completes the proof.

Lemma 3.6 Let μ be any fuzzy subset in an ordered semigroup S. Then $\chi_S \circ \mu (resp., \mu \circ \chi_S)$ is an $(\in, \in \vee q)-$*fuzzy left ideal (resp., right ideal)* of S.

Proof. Let μ be any fuzzy subset in an ordered semigroup S and $x, y \in S$, we have

$$\min\{(\chi_S \circ \mu)(y), 0.5\} = \min\left\{\sup_{y \le ab} \min\{\chi_S(a), \mu(b)\}, 0.5\right\} \le \sup_{xy \le xab}\{\mu(b)\}$$
$$\le \sup_{xy \le cd}\{\mu(d)\} = (\chi_S \circ \mu)(xy)$$

On the other hand, if $y \le x$ then we have

$$\min\{(\chi_S \circ \mu)(x), 0.5\} = \min\left\{\sup_{x \le ab} \min\{\chi_S(a), \mu(b)\}, 0.5\right\}$$
$$\le \sup_{x \le ab}\{\mu(b)\} \le \sup_{y \le cd}\{\mu(d)\} = (\chi_S \circ \mu)(y).$$

Summing up the above statements, $\chi_S \circ \mu$ is an $(\in, \in \vee q)-$*fuzzy left ideal* of S. Similarly, we may prove that $\mu \circ \chi_S$ is an $(\in, \in \vee q)-$*fuzzy right ideal* of S.

4 Regular ordered semigroup

In this section, we present the characterization of regular ordered semigroups in terms of $(\in, \in \vee q)-$*fuzzy left ideals (right ideals, bi-ideals, quasi-ideals interior ideals,)*. We start by formulating the following definition.

Definition 4.1. [20] An ordered semigroup S is said to be regular if for each $a \in S$, there exists $x \in S$ such that $a \leq axa$. Equivalent definitions:

(1) $a \in (aSa], \forall a \in S$;

(2) $A \subseteq (ASA], \forall A \subseteq S$.

Theorem 4.1 Let S be an ordered semigroup. Then the following conditions are equivalent:

(1) Every *bi-ideal* of S is a *left (right) ideal* of S;

(2) Every $(\in, \in \vee q)-$*fuzzy bi-ideal* of S is an $(\in, \in \vee q)-$ fuzzy *left (right) ideal* of S.

Proof. Assume that (1) holds. Let μ. be an $(\in, \in \vee q)-$*fuzzy bi-ideal* of S and $x, y \in S$ be any elements of S. Since the set $(xSx]$ is a *bi-ideal* of S, by the assumption, it is a *left ideal* of S.Thus it follows from the fact S is regular that $xy \in S(ySy] = (S](ySy] \subseteq (ySy]$. Since μ is an $(\in, \in \vee q)-$*fuzzy bi-ideal* of S. Then $\mu(xy) \geq \min\{\mu(yxy), 0.5\} \geq \min\{\mu(y), 0.5\}$.Therefore, μ is an $(\in, \in \vee q)-$ fuzzy left ideal of S. Similarly μ is an $(\in, \in \vee q)-$*fuzzy right ideal* of S, Hence (2) holds.

Conversely, assume that (2) holds. Let B be a *bi-ideal* of S. Then by lemma 3.4, the characteristic function χ_B of B is an $(\in, \in \vee q)-$ fuzzy *bi-ideal* of S. By the assumption, χ_B of B is an $(\in, \in \vee q)-$*fuzzy left ideal* of S and hence B is a *left ideal* of S by Lemma 3.4. Hence (1) holds.

Lemma 4.1. [18] Let S an ordered semigroup. Then S is regular if and only if for every *right ideal* R and every *left ideal* L of S, we have $(RL] = R \bigcap L.$

Theorem 4.2 Let S an ordered semigroup, then the following conditions are equivalent:

(1) S is regular;

(2) $\mu \cap \nu \subseteq \vee q \mu \circ \nu$ for any $(\in, \in \vee q)-$ *fuzzy right ideal* μ and any $(\in, \in \vee q)-$*fuzzy bi-ideal* ν of S;

(3) $\mu \cap \nu \subseteq \vee q \mu \circ \nu$ for any $(\in, \in \vee q)-$fuzzy *bi-ideal* μ and any $(\in, \in \vee q)-$*fuzzy left ideal* ν of S;

(4) $\mu \cap \nu \subseteq \vee q \mu \circ \nu$ for any $(\in, \in \vee q)-$ *fuzzy right ideal* μ and any $(\in, \in \vee q)-$*fuzzy quasi-ideal* ν of S;

(5) $\mu \cap \nu \subseteq \vee q \mu \circ \nu$ for any $(\in, \in \vee q)-$ *fuzzy quasi-ideal* μ and any $(\in, \in \vee q)-$*fuzzy left ideal* ν of S;

(6) $\mu \cap \nu \approx \vee q \mu \circ \nu$ for any $(\in, \in \vee q)-$ *fuzzy right ideal* μ and any $(\in, \in \vee q)-$*fuzzy left ideal* ν of S.

Proof Assume that (1) holds. Let S be an regular ordered semigroup, μ any $(\in, \in \vee q)-$ *fuzzy right ideal* and ν any $(\in, \in \vee q)-$ fuzzy bi-ideal ν of S, respectively. Now let x be any element of S. Since S is regular, there exists $y \in S$ such that $x \le xyx$. Then we have

$$(\mu \circ \nu)(x) = \sup_{x \le ab} \min\{\mu(a), \nu(b)\} \ge \min\{\mu(xy), \nu(x)\}$$

$$\ge \min\{\min\{\mu(x), 0.5\}, \nu(x)\} = \min\{(\mu \cap \nu)(x), 0.5\}$$

This implies $\mu \cap \nu \subseteq \vee q \mu \circ \nu$. Hence (2) holds. And (3) can be similary proved.

On the other hand, it is clear that $(2) \Rightarrow (4) \Rightarrow (6)$ and $(3) \Rightarrow (5) \Rightarrow (6)$ by Lemma 3.5. Now assume that (6) holds. Let R and L be any right ideal and any left ideal of S, respectively. Then by Lemma 3.4, the characteristic functions χ_R and χ_L of R and L are an $(\in, \in \vee q)-$*fuzzy right ideal* and an $(\in, \in \vee q)-$*fuzzy left ideal* of S, respectively. Now, by the assumption and Lemma 2.3, we have

$$\chi_{(RL]} = \chi_R \circ \chi_L \approx \chi_R \cap \chi_L = \chi_{R \cap L}.$$

It follows from Lemma 2.3 that $(RL] = R \cap L$.Therefore S is regular by Lemma 4.1.

Theorem 4.3 Let S be an regular ordered semigroup and μ a fuzzy subset in S. Then μ is an $(\in, \in \vee q)-$*fuzzy ideal* of S if and only if μ is an $(\in, \in \vee q)-$*fuzzy interior ideal* of S.

Proof If μ is an $(\in, \in \vee q)-$ *fuzzy ideal* of S, it is clear that μ is an $(\in, \in \vee q)-$*fuzzy interior ideal* of S. Now let μ is an $(\in, \in \vee q)-$ fuzzy *interior ideal* of S and x any element of S. Since S is regular, there exists $y \in S$ such that $x \leq xyx$.Then we have

$$\mu(xy) \geq \min\{\mu(xyxy), 0.5\} \geq \min\{\min\{\mu(x), 0.5\}, 0.5\} = \min\{\mu(x), 0.5\}$$

In a similar way, we have $\mu(xy) \geq \min\{\mu(y), 0.5\}$. Therefore, μ is an $(\in, \in \vee q)-$ fuzzy ideal of S.

Theorem 4.4 Let S be an regular ordered semigroup and μ a fuzzy subset in S. Then μ is an $(\in, \in \vee q)-$ *fuzzy bi-ideal* of S if and only if μ is an $(\in, \in \vee q)-$*fuzzy quasi- ideal* of S.

Proof. Let μ be an $(\in, \in \vee q)-$ *fuzzy bi-ideal* of S by Lemma3.6, $\chi_S \circ \mu(resp., \mu \circ \chi_S)$ is an $(\in, \in \vee q)-$*left ideal (resp.,right ideal)* of S. By Theorem 4.2, we have

$$\mu \circ \chi_S \cap \chi_S \circ \mu \approx (\mu \circ \chi_S) \circ (\chi_S \circ \mu) = \mu \circ (\chi_S \circ \chi_S) \circ \mu \subseteq \vee q \mu \circ \chi_S \circ \mu \subseteq \vee q \mu .$$

Thus μ is an $(\in, \in \vee q)-$*fuzzy quasi- ideal* of S.

Conversely, let μ is an $(\in, \in \vee q)-$*fuzzy quasi-ideal* of S. Then μ is an $(\in, \in \vee q)-$*fuzzy bi-ideal* of S by Lemma 3.5.

Lemma 4.2[9] Let S an ordered semigroup, then the following conditions are equivalent:

(1) S is regular;

(2) $B = (BSB]$ for every *bi-ideal* B .of S;

(3) $Q = (QSQ]$ for every *quasi-ideal* Q .of S.

Theorem 4.5 Let S an ordered semigroup, then the following conditions are equivalent:

(1) S is regular;

(2) $\mu \approx \mu \circ \chi_S \circ \mu$ for every $(\in, \in \vee q)-$*fuzzy bi-ideal* μ of S;

(3) $\mu \approx \mu \circ \chi_S \circ \mu$ for every $(\in, \in \vee q)-$*fuzzy quasi-ideal* μ of S.

Proof $(1) \Rightarrow (2)$ Assume that (1) holds. Let μ be any $(\in, \in \vee q)-$*fuzzy bi-ideal* of S, and x any element of S. Since S is regular, there exists $y \in S$ such that $x \le xyx$.Then we have

$$
\begin{aligned}
(\mu \circ \chi_S \circ \mu)(x) &= \sup_{x \le ab} \min\{(\mu \circ \chi_S)(a), \mu(b)\} \\
&\ge \min\{(\mu \circ \chi_S)(xy), \mu(x)\} \\
&= \min\{\sup_{xy \le cd} \min\{\mu(c), \mu(d)\}, \mu(x)\} \\
&\ge \min\{\mu(x), \min\{\mu(x), 0.5\}\} \\
&= \min\{\mu(x), 0.5\}.
\end{aligned}
$$

This implies that $\mu \subseteq \vee q \mu \circ \chi_S \circ \mu$. It follows from the fact μ is an $(\in, \in \vee q)-$*fuzzy bi-ideal* that $\mu \circ \chi_S \circ \mu \subseteq \vee q \mu$ and so $\mu \approx \mu \circ \chi_S \circ \mu$.

$(2) \Rightarrow (3)$ This is straightforward by Lemma 3.5.

$(3) \Rightarrow (1)$ Assume that (1) holds. Let Q be any *quasi-ideal* of S. Then by Lemma3.4, the characteristic χ_Q of Q is an $(\in, \in \vee q)-$*fuzzy* $quasi-ideal$ of S. Now, by the assumption and Lemma 2.3, we have

$$\chi_Q \approx \chi_Q \circ \chi_S \circ \chi_Q = \chi_{(QSQ]}.$$

Then it follows from Lemma 2.3 that $Q = (QSQ]$. Therefore, μ is regular by Lemma 4.2.

Theorem 4.6 Let S an ordered semigroup, then the following conditions are equivalent:

(1) S is regular;

(2) $\mu \cap \nu \approx \mu \circ \nu \circ \mu$ for every $(\in, \in \vee q)-$ *fuzzy bi-ideal* μ and every $(\in, \in \vee q)-$*fuzzy interior ideal* ν of S;

(3) $\mu \cap \nu \approx \mu \circ \nu \circ \mu$ for every $(\in, \in \vee q)-$ *fuzzy quasi-ideal* μ and every $(\in, \in \vee q)-$*fuzzy interior ideal* ν of S;

(4) $\mu \cap \nu \approx \mu \circ \nu \circ \mu$ for every $(\in, \in \vee q)-$ *fuzzy bi-ideal* μ and every $(\in, \in \vee q)-$*fuzzy ideal* ν of S;

(5) $\mu \cap \nu \approx \mu \circ \nu \circ \mu$ for every $(\in, \in \vee q)-$ *fuzzy quasi-ideal* μ and every $(\in, \in \vee q)-$*fuzzy ideal* ν of S.

Proof $(1) \Rightarrow (2)$ Assume that (1) holds. Let μ and ν be any $(\in, \in \vee q)-$*fuzzy bi-ideal* and any $(\in, \in \vee q)-$*fuzzy interior ideal* of S, respectively. Then

$$\mu \circ \nu \circ \mu \subseteq \vee q \mu \circ \chi_S \circ \mu \subseteq \vee q \mu$$

and

$$\mu \circ \nu \circ \mu \subseteq \vee q \chi_S \circ \nu \circ \chi_S \subseteq \vee q \nu ,$$

Hence $\mu \circ \nu \circ \mu \subseteq \vee q \mu \cap \nu$. Now let x be any element of S. Since S is regular, there exists $y \in S$ such that $x \leq xyx$.Then we have

$$(\mu \circ \nu \circ \mu)(x) = \sup_{x \leq ab} \min\{(\mu \circ \nu)(a), \mu(b)\}$$

$$\geq \min\{(\mu \circ \nu)(xy), \mu(x)\}$$

$$= \min\{\sup_{xy \leq cd} \min\{\mu(c), \nu(d)\}, \mu(x)\}$$

$$\geq \min\{\min\{\mu(x), 0.5\}, \nu(x)\}$$

$$= \min\{(\mu \cap \nu)(x), 0.5\}.$$

This implies that $\mu \cap \nu \subseteq \vee q \mu \circ \nu \circ \mu$. Therefore $\mu \cap \nu \approx \mu \circ \nu \circ \mu$ and so (2) holds.

It is clear that $(2) \Rightarrow (3) \Rightarrow (5)$ and $(2) \Rightarrow (4) \Rightarrow (5)$. Now, assume that (5) holds. Let μ be any $(\in, \in \vee q)-$ *fuzzy quasi-ideal* of S. Then since χ_S is an $(\in, \in \vee q)-$ *fuzzy ideal* of S, we have $\mu = \mu \cap \chi_S \approx \mu \circ \chi_S \circ \mu$. Therefore, S is regular by Theorem 4.5.

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