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3-Lie bialgebras (L_c, C_d) and (L_c, C_e)

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Abstract

In this paper, we discuss the structure of four dimensional 3-Lie bialgebras of type (L_{c_i}, C_d) and (L_{c_i}, C_e) for $i = 1, 2, 3$. It is proved that there do not exist 3-Lie bialgebras of types (L_{c_i}, C_d) and (L_{c_i}, C_e) for $i = 1, 2$ (Theorem 3.2), and there exist only three classes of 3-Lie bialgebras of types (L_{c_3}, C_d) (Theorem 3,3), and two classes of 3-Lie bialgebras of types (L_{c_3}, C_e) (Theorem 3.4).

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1 Preliminaries

A 3-**Lie algebra** [1] is a vector space L endowed with a linear multiplication $\mu : L^{\wedge 3} \to L$ satisfying that, for all $x, y, z, u, v \in L$,

$\mu(u, v, \mu(x, y, z)) = \mu(x, y, \mu(u, v, z)) + \mu(y, z, \mu(u, v, x)) + \mu(z, x, \mu(u, v, y))$.

For defining 3-Lie coalgebras, we need to define following linear maps

$\omega_i : L \otimes L \otimes L \otimes L \otimes L \to L \otimes L \otimes L \otimes L \otimes L,\ 1 \leq i \leq 3$, by

$\omega_1(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_3 \otimes x_4 \otimes x_1 \otimes x_2 \otimes x_5$,

$\omega_2(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_4 \otimes x_5 \otimes x_1 \otimes x_2 \otimes x_3$,

$\omega_3(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_5 \otimes x_3 \otimes x_1 \otimes x_2 \otimes x_4$.

A 3-Lie coalgebra (L, Δ) [2] is a vector space L with a linear map $\Delta : L \to L \otimes L \otimes L$ satisfying

$Im(\Delta) \subset L \wedge L \wedge L$, and $(1 - \omega_1 - \omega_2 - \omega_3)(1 \otimes 1 \otimes \Delta)\Delta = 0$.

Let (L_1, Δ_1) and (L_2, Δ_2) be 3-Lie coalgebras. If there is a linear isomorphism $\varphi : L_1 \to L_2$ satisfying $(\varphi \otimes \varphi \otimes \varphi)(\Delta_1(e)) = \Delta_2(\varphi(e))$, for all $e \in L_1$, then (L_1, Δ_1) is isomorphic to (L_2, Δ_2), and φ is called *a 3-Lie coalgebra isomorphism*, where $(\varphi\otimes\varphi\otimes\varphi)\sum_i(a_i\otimes b_i\otimes c_i)=\sum_i\varphi(a_i)\otimes\varphi(b_i)\otimes\varphi(c_i)$.

A 3-Lie bialgebra[2] is a triple (L, μ, Δ) such that
(1) (L, μ) is a 3-Lie algebra with the multiplication $\mu: L\wedge L\wedge L\rightarrow L$,
(2) (L, Δ) is a 3-Lie coalgebra with $\Delta: L\rightarrow L\wedge L\wedge L$,
(3) Δ and μ satisfy the following identity, for $x, y, u, v, w\in L$,

$$\Delta\mu(x,y,z)=ad_\mu^{(3)}(x,y)\Delta(z)+ad_\mu^{(3)}(y,z)\Delta(x)+ad_\mu^{(3)}(z,x)\Delta(y),$$

where $ad_\mu^{(3)}(x,y)$, $ad_\mu^{(3)}(z,x)$, $ad_\mu^{(3)}(y,z): L\otimes L\otimes L\rightarrow L\otimes L\otimes L$ are linear maps defined by (similar for $ad_\mu^{(3)}(z,x)$ and $ad_\mu^{(3)}(y,z)$)

$$ad_\mu^{(3)}(x,y)(u\otimes v\otimes w)=(ad_\mu(x,y)\otimes 1\otimes 1)(u\otimes v\otimes w)$$
$$+(1\otimes ad_\mu(x,y)\otimes 1)(u\otimes v\otimes w)+(1\otimes 1\otimes ad_\mu(x,y))(u\otimes v\otimes w)$$
$$=\mu(x,y,u)\otimes v\otimes w+u\otimes\mu(x,y,v)\otimes w+u\otimes v\otimes\mu(x,y,w).$$

Lemma 2.1[1] *Let (L, μ) be a 4-dimensional 3-Lie algebra with $\dim L^1\neq 0, 2$, and e_1, e_2, e_3, e_4 be a basis of L. Then L is isomorphic to one and only one of the following*

$L_{b_1}.\mu(e_2,e_3,e_4)=e_1$. $L_{b_2}.\mu(e_1,e_2,e_3)=e_1$.

L_d. $\mu_d(e_2,e_3,e_4)=e_1, \mu_d(e_1,e_3,e_4)=e_2, \mu_d(e_1,e_2,e_4)=e_3$.

L_e. $\mu_e(e_2,e_3,e_4)=e_1, \mu_e(e_1,e_3,e_4)=e_2, \mu_e(e_1,e_2,e_4)=e_3, \mu_e(e_1,e_2,e_3)=e_4$.

2 3-Lie bialgebras of types (L_c, C_d) and (L_c, C_e)

First We give the classification of 3-Lie coalgebras of the types (L, C_b) and (L, C_c).

Lemma 3.1 [2][4] *Let (L, Δ) be a 4-dimensional 3-Lie coalgebra with m-dimensional derived algebra ($m\geq 3$), and e^1, e^2, e^3, e^4 be a basis of L. Then L isomorphic to one and only one of the, $C_d=(L,\Delta_d)$ and $C_e=(L,\Delta_e)$,*

C_d. $\Delta_d(e^1)=e^2\wedge e^3\wedge e^4$, $\Delta_d(e^2)=e^1\wedge e^3\wedge e^4, \Delta_d(e^3)=e^1\wedge e^2\wedge e^4$;

C_e. $\Delta_e(e^1)=e^2\wedge e^3\wedge e^4$, $\Delta_e(e^2)=e^1\wedge e^3\wedge e^4$, $\Delta_e(e^3)=e^1\wedge e^2\wedge e^4$,
$\Delta_e(e^4)=e^1\wedge e^2\wedge e^3$.

For convenience, in the following, for a 3-Lie bialgebra (L, μ, Δ), if the 3-Lie algebra (L, μ) is the case (L, μ_{c_i}) in Lemma 2.1 and the 3-Lie coalgebra (L, Δ) is the case (L, Δ_d) and (L, Δ_e) in Lemma 3.1, then the 3-Lie bialgebra (L, μ_{c_i}, Δ_d) and (L, μ_{c_i}, Δ_e) are simply denoted by (L_{c_i}, C_d) and (L_{c_i}, C_e), which are called *the 3-Lie bialgebras of type (L_c, C_d), and (L_c, C_e), respectively.*

For a given 3-Lie algebra L, in order to find all the 3-Lie bialgebra structures on L, we should find all the 3-Lie coalgebra structures on L which are compatible with the 3-Lie algebra L. Although a permutation of a basis of L gives isomorphic 3-Lie coalgebra, but it may lead to the non-equivalent 3-Lie bialgebra.

Theorem 3.2 *There do not exist 3-Lie bialgebras of types (L_{c_1}, C_d), (L_{c_1}, C_e), (L_{c_2}, C_d) and (L_{c_2}, C_e).*

Proof By Lemma 3.1 and [5], and a direct computation, we obtain that the following six isomorphic 3-Lie coalgebras of the type C_e :

(1) $\Delta(e_1)=e_2\wedge e_3\wedge e_4, \Delta(e_2)=e_1\wedge e_3\wedge e_4, \Delta(e_3)=e_1\wedge e_2\wedge e_4, \Delta(e_4)=e_1\wedge e_2\wedge e_3$;

(2) $\Delta(e_1) = e_2\wedge e_3\wedge e_4,\ \Delta(e_2) = e_1\wedge e_3\wedge e_4,\ \Delta(e_3) = e_2\wedge e_1\wedge e_4,\ \Delta(e_4) = e_2\wedge e_1\wedge e_3;$
(3) $\Delta(e_1) = e_2\wedge e_3\wedge e_4, \Delta(e_2) = e_3\wedge e_1\wedge e_4, \Delta(e_3) = e_2\wedge e_1\wedge e_4, \Delta(e_4) = e_2\wedge e_3\wedge e_1;$
(4) $\Delta(e_1) = e_2\wedge e_4\wedge e_3,\ \Delta(e_2) = e_1\wedge e_4\wedge e_3,\ \Delta(e_3) = e_2\wedge e_1\wedge e_4,\ \Delta(e_4) = e_2\wedge e_1\wedge e_3;$
(5) $\Delta(e_1) = e_2\wedge e_4\wedge e_3,\ \Delta(e_2) = e_4\wedge e_1\wedge e_3,\ \Delta(e_3) = e_2\wedge e_4\wedge e_1,\ \Delta(e_4) = e_2\wedge e_1\wedge e_3;$
(6) $\Delta(e_1) = e_2\wedge e_4\wedge e_3,\ \Delta(e_2) = e_4\wedge e_3\wedge e_1,\ \Delta(e_3) = e_2\wedge e_4\wedge e_1,\ \Delta(e_4) = e_2\wedge e_3\wedge e_1,$
and twenty-four isomorphic 3-Lie coalgebras of the type C_d:
(1) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_1 \wedge e_3 \wedge e_4,\ \Delta(e_3) = e_1 \wedge e_2 \wedge e_4;$
(2) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_3 \wedge e_1 \wedge e_4,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4;$
(3) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_1 \wedge e_3 \wedge e_4,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4;$
(4) $\Delta(e_1) = e_3 \wedge e_2 \wedge e_4,\ \Delta(e_2) = e_3 \wedge e_1 \wedge e_4,\ \Delta(e_3) = e_1 \wedge e_2 \wedge e_4;$
(5) $\Delta(e_1) = e_3 \wedge e_2 \wedge e_4,\ \Delta(e_2) = e_3 \wedge e_1 \wedge e_4,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4;$
(6) $\Delta(e_1) = e_3 \wedge e_2 \wedge e_4,\ \Delta(e_2) = e_1 \wedge e_3 \wedge e_4,\ \Delta(e_3) = e_1 \wedge e_2 \wedge e_4;$
(7) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_1 \wedge e_4 \wedge e_3,\ \Delta(e_4) = e_2 \wedge e_1 \wedge e_3;$
(8) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_4 \wedge e_1 \wedge e_3,\ \Delta(e_4) = e_2 \wedge e_1 \wedge e_3,$
(9) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_1 \wedge e_4 \wedge e_3,\ \Delta(e_4) = e_1 \wedge e_2 \wedge e_3;$
(10) $\Delta(e_1) = e_4 \wedge e_2 \wedge e_3,\ \Delta(e_2) = e_4 \wedge e_1 \wedge e_3,\ \Delta(e_4) = e_2 \wedge e_1 \wedge e_3;$
(11) $\Delta(e_1) = e_4 \wedge e_2 \wedge e_3,\ \Delta(e_2) = e_1 \wedge e_4 \wedge e_3,\ \Delta(e_4) = e_1 \wedge e_2 \wedge e_3;$
(12) $\Delta(e_1) = e_4 \wedge e_2 \wedge e_3,\ \Delta(e_2) = e_4 \wedge e_1 \wedge e_3,\ \Delta(e_4) = e_1 \wedge e_2 \wedge e_3;$
(13) $\Delta(e_1) = e_3 \wedge e_4 \wedge e_2,\ \Delta(e_3) = e_4 \wedge e_1 \wedge e_2,\ \Delta(e_4) = e_3 \wedge e_1 \wedge e_2;$
(14) $\Delta(e_1) = e_3 \wedge e_4 \wedge e_2,\ \Delta(e_3) = e_1 \wedge e_4 \wedge e_2,\ \Delta(e_4) = e_1 \wedge e_3 \wedge e_2;$
(15) $\Delta(e_1) = e_3 \wedge e_4 \wedge e_2,\ \Delta(e_3) = e_1 \wedge e_4 \wedge e_2,\ \Delta(e_4) = e_3 \wedge e_1 \wedge e_2;$
(16) $\Delta(e_1) = e_4 \wedge e_3 \wedge e_2,\ \Delta(e_3) = e_4 \wedge e_1 \wedge e_2,\ \Delta(e_4) = e_1 \wedge e_3 \wedge e_2;$
(17) $\Delta(e_1) = e_4 \wedge e_3 \wedge e_2,\ \Delta(e_3) = e_4 \wedge e_1 \wedge e_2,\ \Delta(e_4) = e_3 \wedge e_1 \wedge e_2;$
(18) $\Delta(e_1) = e_4 \wedge e_3 \wedge e_2,\ \Delta(e_3) = e_1 \wedge e_4 \wedge e_2,\ \Delta(e_4) = e_1 \wedge e_3 \wedge e_2;$
(19) $\Delta(e_2) = e_4 \wedge e_3 \wedge e_1,\ \Delta(e_3) = e_4 \wedge e_2 \wedge e_1,\ \Delta(e_4) = e_3 \wedge e_2 \wedge e_1;$
(20) $\Delta(e_2) = e_4 \wedge e_3 \wedge e_1,\ \Delta(e_3) = e_2 \wedge e_4 \wedge e_1,\ \Delta(e_4) = e_3 \wedge e_2 \wedge e_1;$
(21) $\Delta(e_2) = e_4 \wedge e_3 \wedge e_1,\ \Delta(e_3) = e_4 \wedge e_2 \wedge e_1,\ \Delta(e_4) = e_2 \wedge e_3 \wedge e_1;$
(22) $\Delta(e_2) = e_3 \wedge e_4 \wedge e_1,\ \Delta(e_3) = e_2 \wedge e_4 \wedge e_1,\ \Delta(e_4) = e_3 \wedge e_2 \wedge e_1;$
(23) $\Delta(e_2) = e_3 \wedge e_4 \wedge e_1,\ \Delta(e_3) = e_2 \wedge e_4 \wedge e_1,\ \Delta(e_4) = e_2 \wedge e_3 \wedge e_1;$
(24) $\Delta(e_2) = e_3 \wedge e_4 \wedge e_1,\ \Delta(e_3) = e_4 \wedge e_2 \wedge e_1,\ \Delta(e_4) = e_3 \wedge e_2 \wedge e_1.$
which are incompatible with the 3-Lie algebra $L_{c_j}, j = 1, 2$. Therefore, there do not exist 3-Lie bialgebras of types (L_{c_i}, C_d), (L_{c_i}, C_e), for $i = 1, 2$. The proof is complete.

Theorem 3.3 *The non-equivalent 3-Lie bialgebras of the type (L_{c_3}, C_d) are as follows:*

$(L_{c_3}, C_d, \Delta_1) : \Delta_1(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta_1(e_2) = e_1 \wedge e_3 \wedge e_4, \Delta_1(e_3) = e_1 \wedge e_2 \wedge e_4;$
$(L_{c_3}, C_d, \Delta_2) : \Delta_2(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta_2(e_2) = e_3 \wedge e_1 \wedge e_4, \Delta_2(e_3) = e_2 \wedge e_1 \wedge e_4;$
$(L_{c_3}, C_d, \Delta_3) : \Delta_3(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta_3(e_3) = e_1 \wedge e_2 \wedge e_4, \Delta_3(e_4) = e_1 \wedge e_2 \wedge e_3.$

Proof From Theorem 3.2, all the twenty-four cases of 3-Lie coalgebras of the type C_d are compatible with the 3-Lie algebra L_{c_3}. And we have 3-Lie bialgebras isomorphisms

(L_{c_3}, C_d) :

$(1) \to (3),\ (2) \to (6),\ (4) \to (5),\ (7) \to (9),\ (8) \to (11), (10) \to (12),$

$(13) \to (24), (14) \to (23), (15) \to (22), (16) \to (21), (17) \to (19)$:

$f(e_1) = e_2, f(e_2) = e_1, f(e_3) = e_3, f(e_4) = e_4;$

(L_{c_3}, C_d) :

$(13) \to (18), (15) \to (16) : f(e_1) = e_1, f(e_2) = -e_2, f(e_3) = e_3, f(e_4) = e_4;$

$(1) \to (10), (2) \to (11), (4) \to (7), (13) \to (14)$:

$f(e_1) = e_1, f(e_2) = e_2, f(e_3) = e_4, f(e_4) = -e_3;$

$(1) \to (4) : f(e_1) = e_2, f(e_2) = -e_1, f(e_3) = e_3, f(e_4) = e_4;$

$(19) \to (20) : f(e_1) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_1, f(e_2) = \frac{-\sqrt{2}}{2}(1 + \sqrt{-1})e_2,$
$f(e_3) = \frac{\sqrt{2}}{2}(1 - \sqrt{-1})e_3, f(e_4) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_4, ;$

$(13) \to (15) : f(e_1) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_1, f(e_2) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_2,$
$f(e_3) = \frac{-\sqrt{2}}{2}(1 - \sqrt{-1})e_3, f(e_4) = \frac{-\sqrt{2}}{2}(1 + \sqrt{-1})e_4;$

$(13) \to (20) : f(e_1) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_2, f(e_2) = \frac{-\sqrt{2}}{2}(1 + \sqrt{-1})e_1,$
$f(e_3) = \frac{\sqrt{2}}{2}(1 + \sqrt{-1})e_4, f(e_4) = \frac{\sqrt{2}}{2}(\sqrt{-1} - 1)e_34.$

Since for any 3-Lie algebra isomorphism

$$h : L_{c_3} \to L_{c_3} : h(e_j) = \sum_{k=1}^{4} a_{jk}e_k, a_{jk} \in F, j = 1, 2, 3, 4, a_{jk} \in F,$$

we have $a_{13} = a_{14} = a_{23} = a_{24} = 0$, and $det \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} = 1$. By a direct computation h is a non-isomorphism of the 3-Lie coalgebra (1) onto 3-Lie coalgebra (2). It follows that (L_{c_3}, C_d, Δ_1) and (L_{c_3}, C_d, Δ_2) are non-equivalent.

Summarizing above discussions, we obtain that the non-equivalent 3-Lie bialgebras are (L_{c_3}, C_d, Δ_1), (L_{c_3}, C_d, Δ_2) and (L_{c_3}, C_d, Δ_3). The proof is complete.

Theorem 3.4 *The non-equivalent 3-Lie bialgebras of the type* (L_{c_3}, C_e) *are as follows:*

$$(L_{c_3}, C_e, \Delta_1) : \ \Delta_1(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta_1(e_2) = e_1 \wedge e_3 \wedge e_4,$$
$$\Delta_1(e_3) = e_1 \wedge e_2 \wedge e_4, \Delta_1(e_4) = e_1 \wedge e_2 \wedge e_3;$$
$$(L_{c_3}, C_e, \Delta_2) : \ \Delta_2(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta_2(e_2) = e_1 \wedge e_4 \wedge e_3,$$
$$\Delta_2(e_3) = e_1 \wedge e_4 \wedge e_2, \Delta_2(e_4) = e_1 \wedge e_2 \wedge e_3.$$

Proof By a direct computation, all the six cases of 3-Lie coalgebras of the type C_{c_e} in the Theorem 3.2 are compatible with the 3-Lie algebra L_{c_3}. And we have isomorphisms of 3-Lie bialgebras

(L_{c_3}, C_e) :

$(1) \to (4) : f(e_1) = -e_1, f(e_2) = e_2, f(e_3) = e_3, f(e_4) = e_4;$

1) $\rightarrow (2), (3) \rightarrow (5), (4) \rightarrow (6) : f(e_1) = e_2, f(e_2) = e_1, f(e_3) = e_3, f(e_4) = e_4$.
For every 3-Lie algebra isomorphism

$$h : L_{c_3} \rightarrow L_{c_3} : h(e_j) = \sum_{k=1}^{4} a_{jk}e_k, a_{jk} \in F, j = 1, 2, 3, 4, a_{jk} \in F,$$

we have $det \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} = 1$. If h is an isomorphism of the 3-Lie coalgebra (1) onto 3-Lie coalgebra (2), then $a_{11} = a_{12} = a_{21} = a_{22} = 0$. This contradicts to the derived algebra $L^1_{c_3} = Fe_1 + Fe_2$. The result follows.

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