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A Modified Non-monotone Method for Generalized Nonlinear Complementarity Problem

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Abstract

In this paper, we use a piecewise NCP function to solve generalized nonlinear complementarity problem. By using a nonmonotone line search, we present a quasi-Newton-type algorithm. Under suitable conditions, the global convergence of the algorithm is proved. In the end, the numerical results are given.

Mathematics Subject Classification: 90C30

Keywords: Generalized nonlinear complementarity problem, piecewise NCP function,quasi-Newton-type method, nonmonotone.

1 Introduction

Consider the following generalized nonlinear complementarity problem (GNCP), finding a vector $x \in R^n$ such that

$$F(x) \geq 0, G(x) \geq 0, F(x)^\top G(x) = 0 \tag{1}$$

where $F, G : R^n \to R^n$ is continuously differentiable function.

The generalized nonlinear complementarity problem has many interesting applications in engineering and economics, and there are several solution methods in recent literature and references [1]-[6]. And in particular, if $G(x) = x$, then the GNCP reduces to the classical nonlinear complementarity problem [7]. There are many methods to solve (1), Wang et al [1] reformulated the GNCP over a polyhedral cone as a system of nonsmooth equations. Then, based on this reformulation, $L - M$ algorithm is employed to obtain its solution. Later, Zhang et al [3] rearranged the GNCP over a polyhedral as a smoothing system of equations, and then developed a smoothing Newton-type method for solving it. They proved that their suggested method has local superlinear (quadratic)

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convergence rate under certain conditions. But one has to do lots of computations to decide whether the linear system is solvable or not, which is in Step 2 of the algorithm presented in [3]. Moreover, A Broyden-like algorithm also received much attention in solving GNCP. Chen et al [8] reformulated the GNCP over a polyhedral cone as a smoothing system of equations and then suggest a smoothing Broyden-like method for solving it. In their algorithm, only one line search to perform and update only one matrix per iteration.

Motivated by the above ideas, in this paper we present a smooth quasi-Newton-like algorithm for the GNCP based on the 3-1 piecewise NCP function. And the algorithm makes use of the nonmonotone line search rule. Moreover, we transform the generalized nonlinear complementarity problem to smooth equations so that only one smooth equation needs to be solved at each iteration. We will show that this algorithm possesses some nice global convergence properties under mild assumptions.

This paper is organized as follows. In the next section, we introduce the piecewise linear NCP function. In Section 3, we introduce the improved algorithm in detail. In Section 4, we prove the algorithm to be well-defined and present the convergent properties of the presented algorithm. Numerical results are reported in Section 5.

2 Preliminary Notes

We give some definitions: P_0-function, NCP pair, NCP function.

Definition 2.1 *$F : R^n \longrightarrow R^n$ is a continuously differetiable P_0-function, i.e., for all $x, y \in R^n$ with $x \neq y$, there exists an index i_0 such that*

$$x_{i_0} \neq y_{i_0}, (x_{i_0} - y_{i_0})[F_{i_0}(x) - F_{i_0}(y)] \geq 0 \tag{2}$$

Definition 2.2 *We call a pair $(a, b) \in R^2$ to be an NCP pair if $a \geq 0, b \geq 0$ and $ab = 0$; a function $\Phi : R^2 \longrightarrow R$ is called an NCP function if $\Phi(a, b) = 0$ if and only if (a, b) is an NCP pair.*

The 3-1 piecewise linear NCP function is defined as:

Definition 2.3

$$\Phi(a,b) = \begin{cases} 3a - \dfrac{a^2}{b} & \text{if } b \geq a > 0, \text{ or } 3b > -a \geq 0 \ , \\ 3b - \dfrac{b^2}{a} & \text{if } a > b > 0, \text{ or } 3a > -b \geq 0 \ , \\ 9a + 9b & \text{if } a \leq 0 \text{ and } -a \geq 3b, \text{ or } b \leq -3a \leq 0 \ . \end{cases} \tag{3}$$

If $(a,b)\neq(0,0)$,then

$$\bigtriangledown\Phi(a,b)=\begin{cases}\begin{pmatrix}3-\dfrac{2a}{b}\\ \dfrac{a^2}{b^2}\end{pmatrix} & \text{if } b\geq a>0, \text{ or } 3b>-a\geq 0\ ,\\ \begin{pmatrix}\dfrac{b^2}{a^2}\\ 3-\dfrac{2b}{a}\end{pmatrix} & \text{if } a>b>0, \text{ or } 3a>-b\geq 0\ ,\\ \begin{pmatrix}9\\ 9\end{pmatrix} & \text{if } 0\geq a \text{ and } -a\geq 3b, \text{ or } b\leq -3a\leq 0\ .\end{cases} \tag{4}$$

Denote $H: R^{3n}\longrightarrow R^{3n}$

$$H(s,t,x)=\begin{pmatrix}s-F(x)\\ t-G(x)\\ \Phi(s,t)\end{pmatrix}, \tag{5}$$

where s approaches $F(x)$, t approaches $G(x)$ and $s=F(x)$, $t=G(x)$ at the solution. $\Phi(s,t)$ is 3-1 piecewise linear NCP function.

Therefore, we can reform the GNCP(1) to the problem:

$$\min\ \Psi(s,t.x)=\|H(s,t,x)\| \tag{6}$$

3 A Non-monotone quasi-Newton-type algorithm

we need to introduce the following symbols,

$$(\xi_i^k,\eta_i^k)=\begin{cases}(1,1) & \text{(x,s)=(0,0)}\\ \nabla\Phi(x,s) & \text{otherwise}\end{cases} \tag{7}$$

$i=1,2\cdots n$, obviously, $\xi_i^k>0$ and $\eta_i^k>0$.

Compute the Jacobian matrix $V(s^k,t^k,x^k)$ of $H(s^k,t^k,x^k)$, we get

$$V(s^k,t^k,x^k)=\begin{pmatrix}I & 0 & -\nabla F(x^k)\\ 0 & I & -\nabla G(x^k)\\ diag(\xi_i^k) & diag(\eta_i^k) & 0\end{pmatrix} \tag{8}$$

where I is identity matrix of $n\times n$, $\nabla F(x^k),\nabla G(x^k)$ are gradient matrix of $F(x^k)$ and $G(x^k)$ respectively, $diag(\xi_i^k)$ or $diag(\eta_i^k)$ denotes the diagonal matrix whose ith diagonal element is ξ_i^k or η_i^k respectively.

Algorithm 3.1

Step0 Initialization:

Given initial point $(s^0, t^0, x^0) \in R^{3n}$, $\tau \in (0,1)$, $0 < \theta, \overline{\theta} < 1$, $B_0 = V(s^0, t^0, x^0)$, $k = 0$.

Step1 If $\Psi(s^k, t^k, x^k) = 0$ then stop. Otherwise, calculation of the search direction:

Calculate u^k, v^k and d^k by solving the following linear system in (u, v, d):

$$B_k \begin{pmatrix} u \\ v \\ d \end{pmatrix} = \begin{pmatrix} F(x^k) - s^k \\ G(x^k) - t^k \\ -\Phi(s^k, t^k) \end{pmatrix} \tag{9}$$

Step2 Nonmonotone line search.

Step2.1 if

$$\Psi(s^k + u^k, t^k + v^k, x^k + d^k) \leq \theta\Psi(s^k, t^k, x^k) \tag{10}$$

$$\|\Phi(s^k + u^k, t^k + v^k)\| \leq \theta \max_{0 \leq r \leq m(k)-1} \|\Phi^{k-r}\| \tag{11}$$

where $m(0) = 0$, $0 \leq m(k) \leq \min\{m(k-1)+1, M\}$, M is a positive constant. then let

$$s^{k+1} = s^k + u^k, t^{k+1} = t^k + v^k, x^{k+1} = x^k + d^k \tag{12}$$

go to step 3, otherwise go to step 2.2.

Step2.2 let

$$s^{k+1} = s^k + \alpha_k u^k, t^{k+1} = t^k + \alpha_k v^k, x^{k+1} = x^k + \alpha_k d^k \tag{13}$$

where $\alpha_k = \tau^j$ $(0 < \tau < 1)$ and j is the smallest non-negative integer and satisfied:

$$\Psi(s^{k+1}, t^{k+1}, x^{k+1}) \leq \theta\Psi(s^k, t^k, x^k) \tag{14}$$

$$\|\Phi(s^{k+1}, t^{k+1})\| \leq \theta \max_{0 \leq r \leq m(k)-1} \|\Phi^{k-r}\| \tag{15}$$

Step3 Update B_k to get B_{k+1}:

$$B_{k+1} = B_k + \theta_k \frac{(y^k - B_k z^k)(z^k)^\top}{\|z^k\|^2} \tag{16}$$

where

$$z^k = \begin{pmatrix} s^{k+1} \\ t^{k+1} \\ x^{k+1} \end{pmatrix} - \begin{pmatrix} s^k \\ t^k \\ x^k \end{pmatrix}, y^k = H(s^{k+1}, t^{k+1}, x^{k+1}) - H(s^k, t^k, x^k). \tag{17}$$

select θ_k to satisfy $|\theta_k - 1| \leq \overline{\theta}$ and matrix B_{k+1} is nonsingular.

Step4 Let k=k+1 and go to Step 1.

4 Convergent properties

In order to achieve the convergence of the algorithm, we give some Assumptions as follows:

Assumption 4.1

(a): $F, G : R^n \to R^n$ is continuously differentiable P_0-function.

(b): For initial point $(s^0, t^0, x^0) \in R^{3n}$, denote the level set of $\Psi(s,t,x)$ by

$$L(s^0, t^0, x^0) = \{(s,t,x) \in R^{3n} | \Psi(s,t,x) \leq \Psi(s^0, t^0, x^0)\} \tag{18}$$

by the definition, the level set of $\Psi(s,t,x)$ is bounded.

Remark 4.1 $F(x)$, $G(x)$ are P_0-function, then $\nabla F(x)$, $\nabla G(x)$ is positive semidefinite.

Lemma 4.1 *If $H(s^0, t^0, x^0) \neq 0$ then $B_0 = V(s^0, t^0, x^0)$ is nonsingular.*

Proof: Assume $H(s^0, t^0, x^0) \neq 0$. If $V(s^0, t^0, x^0) \cdot (u, v, d)^\top = 0$ for some $(u, v, d) \in R^{3n}$, where $u = (u_1 \cdots u_n)^\top$, $v = (v_1 \cdots v_n)^\top$, $d = (d_1 \cdots d_n)^\top$ then

$$Iu - \nabla F(x^0)d = 0 \tag{19}$$

$$Iv - \nabla G(x^0)d = 0 \tag{20}$$

$$diag(\xi^0)u + diag(\eta^0)v = 0 \tag{21}$$

From the definitions of ξ_i^0 and η_i^0 ,we know that $\xi_i^0 > 0$ and $\eta_i^0 > 0$ for all i, $i = 1 \cdots n$. So, we have

$$v = -(diag(\eta^0))^{-1} diag(\xi^0)u \tag{22}$$

then

$$d = -(\nabla G(x^0))^{-1}(diag(\eta^0))^{-1} diag(\xi^0)u \tag{23}$$

Substitute d in (19) by (23), and multiplying by $u^\top$, we have

$$u^\top Iu + u^\top \nabla F(x^0)(\nabla G(x^0))^{-1}(diag(\eta^0))^{-1} diag(\xi^0)u = 0 \tag{24}$$

The fact that $F(x)$, $G(x)$ is the P_0-function, so all principal minor determinant of the $\nabla F(x)$, $\nabla G(x)$ is non-negative, that is to say, $\nabla F(x^0)$, $(\nabla G(x^0))^{-1}$ is positive semidefinite. And matrix $(diag(\eta^0))^{-1} diag(\xi^0)$ is positive definite. So $u = 0$. also $v = 0$, $d = 0$. Hence, B_0 is nonsingular.

Lemma 4.2 $\Phi(s^k, t^k) \longrightarrow 0$, *as* $k \longrightarrow \infty$.

Proof: In view of convenience, if for all sufficiently large k (10),(11) holds, define

$\|\Phi^{l(k)}\| = \max_{0\le r\le m(k)-1} \|\Phi^{k-r}\|$, where $k - m(k) + 1 \le l(k) \le k$.

Since $m(k+1) \le m(k) + 1$, then

$$\begin{aligned}\|\Phi^{l(k+1)}\| &= \max_{0\le r\le m(k+1)-1} \|\Phi^{k+1-r}\| \\ &\le \max_{0\le r\le m(k)} \|\Phi^{k+1-r}\| \\ &= \max\{\|\Phi^{l(k)}\|, \|\Phi^{k+1}\|\} \\ &= \|\Phi^{l(k)}\|\end{aligned}$$

So, $\|\Phi^{l(k)}\|$ is monotone decreasing, which implies that the $\{\|\Phi^{l(k)}\|\}$ converges.
It follows from (10),(11) that $\|\Phi^{l(k)}\| \le \theta\|\Phi^{l(l(k)-1)}\|$.
Since $\theta \in (0,1)$, therefore $\{\|\Phi^{l(k)}\|\} \longrightarrow 0(k \longrightarrow \infty)$.
Therefore $\|\Phi^{k+1}\| \le \theta\|\Phi^{l(k)}\| \longrightarrow 0(k \longrightarrow \infty)$ holds by the Algorithm.
That is, $\lim_{k\longrightarrow\infty} \|\Phi^k\| = 0$.

Lemma 4.3 *$u^k \longrightarrow 0,\ v^k \longrightarrow 0,\ d^k \longrightarrow 0,\ H(s^k, t^k, x^k) \longrightarrow 0, (k \longrightarrow \infty)$.*

Proof: By Algorithm, we known that $\Psi(s^k, t^k, x^k)$ is monotone decreasing. And $\Psi(s, t, x) = \|H(s, t, x)\|$, so $\Psi(s, t, x) \ge 0$. Therefore, $\Psi(s^k, t^k, x^k) \to 0$, as $k \to \infty$. Namely, $H(s^k, t^k, x^k) \to 0$, as $k \to \infty$.
Therefore,

$$B_k \begin{pmatrix} u^k \\ v^k \\ d^k \end{pmatrix} = \begin{pmatrix} F(x^k) - s^k \\ G(x^k) - t^k \\ -\Phi(s^k, t^k) \end{pmatrix} \longrightarrow 0, as\ k \longrightarrow \infty \tag{25}$$

so B_k is nonsingular by Algorithm. The conclusion follows.

Lemma 4.4 *let $\{(s^k, t^k, x^k)\}$ is generated sequence by Algorithm, then $\{(s^k, t^k, x^k)\} \subset L(s^0, t^0, x^0)$.*

Proof: We prove it by induction, when k=0, it is clearly that $(s^0, t^0, x^0) \in L(s^0, t^0, x^0)$. Assume $(s^k, t^k, x^k) \in L(s^0, t^0, x^0)$, then we have $\Psi(s^k, t^k, x^k) \le \Psi(s^0, t^0, x^0)$. By (3.3) and (3.7), we get $\Psi(s^{k+1}, t^{k+1}, x^{k+1}) \le \theta\Psi(s^k, t^k, x^k)$, $0 < \theta < 1$. So, $(s^{k+1}, t^{k+1}, x^{k+1}) \in L(s^0, t^0, x^0)$. we obtain $\{(s^k, t^k, x^k)\} \subset L(s^0, t^0, x^0)$ for all k.

Theorem 4.5 *Suppose the Assumption 4.1 holds, let $\{(s^k, t^k, x^k)\}$ be generated by Algorithm, (s^*, t^*, x^*) is an accumulation point of the $\{(s^k, t^k, x^k)\}$, then x^* is a solution of (1).*

Proof: From above Lemmas, we know $\{(s^k, t^k, x^k)\} \subset L(s^0, t^0, x^0)$. By Assumption, we know $L(s^0, t^0, x^0)$ is bounded. So, $\{(s^k, t^k, x^k)\}$ has convergence column. Thus, suppose $\{(s^k, t^k, x^k)\}_{k\in K}$ convergence to (s^*, t^*, x^*). Then we only need to prove $H(s^*, t^*, x^*) = 0$. Because the subsequence of convergence sequence is convergence, and them limitation are same. We only need to take the subsequence of $\{(s^k, t^k, x^k)\}_{k\in K}$ in the following proof.

In the Algorithm, there are two types of successive iteration, we have $\Psi(s^k + u^k, t^k + v^k, x^k + d^k) \leq \theta\Psi(s^k, t^k, x^k)$, and $\Psi(s^k + \alpha_k u^k, t^k + \alpha_k v^k, x^k + \alpha_k d^k) \leq \theta\Psi(s^k, t^k, x^k)$ So, $\Psi(s^{k+1}, t^{k+1}, x^{k+1}) \leq \theta\Psi(s^k, t^k, x^k)$ for all k. This suggests that $\liminf\limits_{k\to\infty} \Psi(s^k, t^k, x^k) = 0$, for all k.

5 Numerical Results

In this section, let the parameters used in Algorithm 3.1 be $\theta = 0.9$, $\tau = 0.8$, $\theta_k \equiv 1$. We used $\|H(s,t)\| \leq 10^{-6}$ as the stop criterion.

In the following tables, ST denotes the initial point. IT,Fv denotes the number of iterations and the value of $\|H(s,t)\|$. VC denotes the value of $F(x)^\top G(x)$ at the final iteration. y^* is an accurate solution of the GNCP, y^k is the final value of y.

Example 5.1 : We consider the following problem: find $y \in R^n$ such that
$y - m(y) \geq 0, F(y) \geq 0, F(y)^\top (y - m(y)) = 0$
where $m_i : R^n \longrightarrow R, i = 1, \ldots, n$, and

$$F(y) = Ay + b = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} y + \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \tag{26}$$

and $m(y) = \psi(Ay + b)$ with $\psi : R^n \longrightarrow R^n$ being twice continously differentiable. The following choices of function ψ define our test problems.

$$P1.\ \psi_i(x) = -0.5 - x_i, \qquad i = 1, 2, \ldots, n. \tag{27}$$

For each problem, the following starting point were used, namely,
$(a)(0, 0, \ldots, 0)^\top$ $(b)(-0.5, -0.5, \ldots, -0.5)^\top$
$(c)(-1, -1, \ldots, -1)^\top$ $(d)(0.5, 0.5, \ldots, 0.5)^\top$

We test this problem by using $s^0 = (2, 2, \cdots, 2)^\top$, $t^0 = (1, 1, \cdots, 1)^\top$, and choose n=4 as the dimension of the problem. The results are listed in Table 1.

Table 1

P	ST	IT	FV	VC	y^k
(1)	(a)	7	7.1511×10^{-9}	5.1674×10^{-11}	[-0.9000;-1.2000;-1.2000;-0.9000]
(1)	(b)	7	7.0881×10^{-9}	5.9212×10^{-11}	[-0.9000;-1.2000;-1.2000;-0.9000]
(1)	(c)	7	7.4632×10^{-9}	6.4147×10^{-11}	[-0.9000;-1.2000;-1.2000;-0.9000]
(1)	(d)	7	7.4449×10^{-9}	4.3764×10^{-11}	[-0.9000;-1.2000;-1.2000;-0.9000]

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