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Global boundedness for solutions to some anisotropic elliptic systems

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Abstract

This paper deals with anisotropic solutions $u \in u_* + W_0^{1,(p_i)}(w, \Omega, \mathrm{R}^N)$ to the nonlinear elliptic system

$$-\sum_{i=1}^{n} D_i(a_i^\alpha(x, Du(x))) = 0, \quad \alpha = 1, \cdots, N.$$

We present a monotonicity inequality on the matrix $a = (a_i^\alpha) \in \mathrm{R}^{N \times n}$ with weight, which guarantees global pointwise bounds for the anisotropic solutions u with gradient constraint.

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1 Introduction and Statement of results.

Throughout this paper Ω will stand for a bounded open domain in R^n, $n \geq 2$. For $p_1, \cdots, p_n \in (1, +\infty)$, we let $\bar{p} = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i} \right)^{-1}$ and $p_i' = \frac{p_i}{p_i - 1}$ be the harmonic mean of $p_1, \cdots, p_n$ and the Hölder conjugate of p_i, respectively.

Let w_i $(i = 1 \cdots, n)$ to be functions in Ω such that $w_i > 0$ a.e. in Ω, and

$$w_i \in L^1_{loc}(\Omega), \quad \frac{1}{w_i} \in L^{1/(p_i - 1)}(\Omega). \tag{1.1}$$

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Denote

$$W^{1,(p_i)}(w,\Omega)=\{u\in L^1(\Omega): w_i|D_iu|^{p_i}\in L^1(\Omega),\ i=1,\cdots,n\}.$$

The norm for $u\in W^{1,(p_i)}(w,\Omega)$ is defined by

$$\|u\|_{1,(p_i),w}=\int_\Omega |u|dx+\sum_{i=1}^n\left(\int_\Omega w_i|D_iu|^{p_i}dx\right)^{1/p_i}.$$

By the second inclusion of (1.1), the set $W^{1,(p_i)}(w,\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{1,(p_i),w}$. By virtue of the first inclusion of (1.1), we have $C_0^\infty(\Omega)\subset W^{1,(p_i)}(w,\Omega)$. We denote by $W_0^{1,(p_i)}(w,\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in the norm of $W^{1,(p_i)}(w,\Omega)$. The set $W_0^{1,(p_i)}(w,\Omega)$ is a reflexive Banach space with respect to the norm induced by $\|\cdot\|_{1,(p_i),w}$. We denote by $W^{1,(p_i)}(w,\Omega,\mathrm{R}^N)$ the set of all vector valued functions $u=(u^1,\cdots,u^N)$ such that for every $j\in\{1,\cdots,N\}$ we have $u^j\in W^{1,(p_i)}(w,\Omega)$. In particular, $W^{1,(p_i)}(\Omega)$, $W_0^{1,(p_i)}(\Omega)$, $W^{1,(p_i)}(\Omega,\mathrm{R}^N)$ and $W_0^{1,(p_i)}(\Omega,\mathrm{R}^N)$ stand for the special cases of $W^{1,(p_i)}(w,\Omega)$, $W_0^{1,(p_i)}(w,\Omega)$, $W^{1,(p_i)}(w,\Omega,\mathrm{R}^N)$ and $W_0^{1,(p_i)}(w,\Omega,\mathrm{R}^N)$ with $w_i\equiv 1$, $i=1,\cdots,n$, respectively.

For a vector $m=(m_1,\cdots,m_n)\in\mathrm{R}^n$ with $m_i>0$, $i=1,\cdots,n$, we set

$$q_m=n\left(\sum_{i=1}^n\frac{1+m_i}{m_ip_i}-1\right)^{-1}.$$

We let $\varphi:\Omega\to\mathrm{R}$ be a nonnegative function and $u_*\in W^{1,(p_i)}(w,\Omega,\mathrm{R}^N)$ be such that $|Du_*(x)|\le\varphi(x)$, a.e. Ω. We define

$$\mathcal{C}_{u_*,\varphi}=\{v\in u_*+W_0^{1,(p_i)}(w,\Omega,\mathrm{R}^N): |Dv(x)|\le\varphi(x)\text{ for a.e. }x\in\Omega\}.$$

It is obvious that $u_*\in\mathcal{C}_{u_*,\varphi}$. Therefore, $\mathcal{C}_{u_*,\varphi}\ne\emptyset$. It is easy to see that the set $\mathcal{C}_{u_*,\varphi}$ is convex and closed in $W^{1,(p_i)}(w,\Omega,\mathrm{R}^N)$.

We consider the nonlinear elliptic system

$$-\sum_{i=1}^n D_i(a_i^\alpha(x,Du(x)))=0,\quad \alpha=1,\cdots,N.$$

where $Du=(D_1u,\cdots,D_nu)$ denotes the gradient of a vector valued function $u=(u^1,\cdots,u^N):\Omega\to\mathrm{R}^N$, and $a=(a_i^\alpha):\Omega\times\mathrm{R}^{N\times n}\to\mathrm{R}^{N\times n}$ is a Carathéodory $N\times n$ matrix, that is, $a_i^\alpha(x,\xi)$ is measurable with respect to x and continuous with respect to ξ, $i=1,\cdots,n$, $\alpha=1,\cdots,N$.

We assume that there exist a constant $\nu>0$ and a function $M(x)\in L^r(\Omega)$, $r>1$, such that

$$\nu\sum_{i=1}^n w_i|A_i-\tilde{A}_i|^{p_i}\le\sum_{i=1}^n\sum_{\alpha=1}^N a_i^\alpha(x,A)(A_i^\alpha-\tilde{A}_i^\alpha)+M(x),\tag{1.2}$$

holds true for every pair of matrices $A, \tilde{A} \in \mathrm{R}^{N\times n}$ such that there exists a row β with $\tilde{A}^\beta = 0$ and for every remaining row $\alpha \neq \beta$ we have $\tilde{A}^\alpha = A^\alpha$.

In order to make finite the integrals over subsets of Ω of the first term in the right hand side of (1.2), we assume that

$$|a_i^\alpha(x, A)| \leq c\left(1 + \sum_{j=1}^n |A_j|^{p_j}\right)^{1/p_i'}.$$

The main result of this paper is the following theorem.

Theorem 1.1 *Let $m \in R^n$, and let the following two conditions be satisfied:*
(a) for every $i \in \{1, \cdots, n\}$ we have $m_i \geq 1/(p_i - 1)$ and $1/w_i \in L^{m_i}(\Omega)$;
(b) $q_m > \bar{p}$.
We let $u \in \mathcal{C}_{u_*,\varphi}$ be such that

$$\forall v \in \mathcal{C}_{u_*,\varphi}, \ \int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N a_i^\alpha(x, Du) D_i(u^\alpha - v^\alpha) dx \leq 0. \tag{1.3}$$

Then, for every component u^β of u, we have

$$\inf_{\partial\Omega} u_*^\beta(x) - c_* \leq u^\beta(x) \leq \sup_{\partial\Omega} u_*^\beta(x) + c_* \tag{1.4}$$

for almost every $x \in \Omega$, where

$$c_* = c\left(\frac{\|M\|_{L^r(\Omega)}}{\nu}\right)^{\frac{1}{\bar{p}}} |\Omega|^{\left[(1-\frac{1}{r})\frac{q_m}{\bar{p}}-1\right]\frac{1}{q_m}} 2^{(1-\frac{1}{r})\frac{q_m}{\bar{p}}\left[(1-\frac{1}{r})\frac{q_m}{\bar{p}}-1\right]^{-1}},$$

where $|\Omega|$ is the n-dimensional Lebesgue measure of Ω, and c and ν are the constants from (2.1) and (1.2), respectively.

Remark 1.2 *We refer the readers to [1-6] for some related results.*

A model density f for the monotonicity inequality (1.2) is given in the following.

Theorem 1.3 *For every $i = 1, \cdots, n$, let us consider $p_i \geq 2$ and $t_i > 0$. For $A \in R^{N\times n}$ we consider $a(x, A) = (a_1(x, A), \cdots, a_n(x, A))$ with*

$$a_i(x, A) = t_i w_i |A_i|^{p_i - 2} A_i + \frac{m(x)}{1 + \|A\|}, \quad i = 1, \cdots, n,$$

where $m(x) \geq 0$. Then the inequality (1.2) holds true with

$$\nu = \min_{1\leq i\leq n} \{t_i\} \quad \text{and} \quad M(x) = m(x).$$

2 Proof of Theorems 1.2 and 1.3.

In order to prove Theorem 1, we need two preliminary lemmas.

The following lemma is the Sobolev Imbedding Theorem with weight, which comes from [7, Proposition 2.1], the proof can be found in [8].

Lemma 2.1 *Let $m \in R^n$, and let the following conditions be satisfied: for every $i \in \{1, \cdots, n\}$ we have $m_i \geq 1/(p_i - 1)$ and $1/w_i \in L^{m_i}(\Omega)$. Then $W_0^{1,(p_i)}(w, \Omega) \subset L^{q_m}(\Omega)$, and there exists a positive constant c such that for every function $v \in W_0^{1,(p_i)}(w, \Omega)$,*

$$\left(\int_\Omega |v|^{q_m} dx\right)^{1/q_m} \leq c \prod_{i=1}^n \left(\int_\Omega w_i |D_i v|^{p_i} dx\right)^{1/np_i}. \tag{2.1}$$

The next lemma comes from [9, Lemma 4.1].

Lemma 2.2 *Let $\chi : [t_0, +\infty) \to [0, +\infty)$ be non-increasing. We assume that there exist $\tilde{C}, a > 0$ and $b > 1$ such that*

$$t_0 \leq t < T \Rightarrow \chi(T) \leq \frac{\tilde{C}}{(T-t)^a} [\chi(t)]^b.$$

Then it results that

$$\chi(t_0 + d) = 0,$$

where

$$d = \left[\tilde{C} \left(\chi(t_0)\right)^{b-1} 2^{\frac{ab}{b-1}}\right]^{\frac{1}{a}}.$$

Proof of Theorem 1.1. As in the proof of Lemma 2.1 in [1], we define $I_{\beta,t} : \mathrm{R}^N \to \mathrm{R}^N$ as follows:

$$\forall y = (y^1, \cdots, y^N) \in \mathrm{R}^N, \quad I_{\beta,t}(y) = (I^1_{\beta,t}(y), I^2_{\beta,t}(y), \cdots, I^N_{\beta,t}(y))$$

with

$$I^\alpha_{\beta,t}(y) = \begin{cases} y^\alpha, & \alpha \neq \beta \\ y^\beta \wedge t = \min\{y^\beta, t\}, & \alpha = \beta. \end{cases}$$

We may assume $\sup_{\partial\Omega} u^\beta_* < t_0 \leq t$, since otherwise, $\sup_{\partial\Omega} u^\beta_* = +\infty$, the right hand side inequality of (1.4) holds true trivially. For such t and $u \in \mathcal{C}_{u_*,\varphi}$, we need to show that $I_{\beta,t}(u) \in \mathcal{C}_{u_*,\varphi}$. In fact, it is obvious that $I_{\beta,t}(u) \in u_* + W_0^{1,(p_i)}(w, \Omega, \mathrm{R}^N)$; in order to show that $|DI_{\beta,t}(u)(x)| \leq K(x)$ we notice that

$$D_i(I^\alpha_{\beta,t}(u)) = \begin{cases} D_i u^\alpha, & \alpha \neq \beta, \\ D_i u^\beta 1_{\{u^\beta \leq t\}}, & \alpha = \beta, \end{cases} \tag{2.1}$$

where 1_B is the characteristic function of the set B, that is, $1_B(x) = 1$ if $x \in B$ and $1_B(x) = 0$ otherwise. In both cases of (2.1), we have $|DI_{\beta,t}(u)(x)| \leq K(x)$.

(1.3) with $v = I_{\beta,t}(u)$ acting as a test function yields

$$\int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N a_i^\alpha(x, Du) D_i(u^\alpha - I_{\beta,t}^\alpha(u))dx \leq 0. \tag{2.2}$$

It is obvious that

$$D_i(u^\alpha - I_{\beta,t}^\alpha(u)) = \begin{cases} 0, & \alpha \neq \beta, \\ D_i u^\beta \cdot 1_{\{u^\beta > t\}} & \alpha = \beta. \end{cases}$$

On $\{x \in \Omega : u^\beta > t\}$ we have $D(I_{\beta,t}^\beta(u)) = 0$ and, for $\alpha \neq \beta$, $D(I_{\beta,t}^\alpha(u)) = Du^\alpha$; so we can apply (1.2) with $A = Du$ and $\tilde{A} = D(I_{\beta,t}(u))$; we obtain that

$$\nu \sum_{i=1}^n w_i |D_i u - D_i(I_{\beta,t}(u))|^{p_i} \leq \sum_{i=1}^n \sum_{\alpha=1}^N a_i^\alpha(x, Du) \left(D_i u^\alpha - D_i \left(I_{\beta,t}^\alpha(u)\right)\right) + M(x) \tag{2.3}$$

for $x \in \{x \in \Omega : u^\beta > t\}$. On $\{x \in \Omega : u^\beta \leq t\}$, $D(I_{\beta,t}(u)) = Du$, thus

$$\nu \sum_{i=1}^n w_i |D_i u - D_i(I_{\beta,t}(u))|^{p_i} = \sum_{i=1}^n \sum_{\alpha=1}^N a_i(x, Du) \left(D_i u^\alpha - D_i \left(I_{\beta,t}^\alpha(u)\right)\right) = 0, \tag{2.4}$$

for $x \in \{x \in \Omega : u^\beta \leq t\}$. From (2.3) and (2.4) we have

$$\nu \sum_{i=1}^n w_i |D_i u - D_i(I_{\beta,t}(u))|^{p_i} \leq \sum_{i=1}^n \sum_{\alpha=1}^N a_i(x, Du) \left(D_i u^\alpha - D_i \left(I_{\beta,t}^\alpha(u)\right)\right) + M(x) \cdot 1_{\{u^\beta > t\}},$$

from which we derive

$$\begin{aligned} &\nu \int_\Omega \sum_{i=1}^n w_i |D_i u - D_i(I_{\beta,t}(u))|^{p_i} dx \\ \leq \ &\int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N a_i^\alpha(x, Du) \left(D_i u^\alpha - D_i \left(I_{\beta,t}^\alpha(u)\right)\right) dx + \int_{\{u^\beta > t\}} M(x) dx. \end{aligned}$$

We apply (2.2) and we get

$$\nu \int_\Omega \sum_{i=1}^n w_i |D_i u - D_i(I_{\beta,t}(u))|^{p_i} dx \leq \int_{\{u^\beta > t\}} M(x) dx. \tag{2.5}$$

If we define $\phi = (u^\beta - t) \vee 0 = \max\{u^\beta - t, 0\}$, then (2.5) together with Hölder inequality yields

$$\nu \sum_{i=1}^n \int_\Omega w_i |D_i \phi|^{p_i} dx \leq \int_{\{u^\beta > t\}} M(x) dx \leq \|M\|_{L^r(\Omega)} |\{u^\beta > t\}|^{1-\frac{1}{r}}.$$

If $r < +\infty$, we apply Hölder inequality and we get

$$\int_{\{u^\beta > t\}} M(x)dx \le \|M\|_{L^r(\Omega)} |\{u^\beta > t\}|^{1-\frac{1}{r}}.$$

If $r = +\infty$, then

$$\int_{\{u^\beta > t\}} M(x)dx \le \|M\|_{L^\infty(\Omega)} |\{u^\beta > t\}| = \|M\|_{L^r(\Omega)} |\{u^\beta > t\}|^{1-\frac{1}{r}}.$$

In both cases, from (2.5) it results that

$$\sum_{i=1}^n \int_\Omega w_i |D_i\phi|^{p_i} dx \le \frac{\|M\|_{L^r(\Omega)}}{\nu} |\{u^\beta > t\}|^{1-\frac{1}{r}}.$$

We apply Lemma 2.1 and we get

$$\begin{aligned} & \left(\int_{\{u^\beta > t\}} |u^\beta - t|^{q_m} dx \right)^{1/q_m} \\ = & \left(\int_{\{u^\beta > t\}} |\phi|^{q_m} dx \right)^{1/q_m} = \left(\int_\Omega |\phi|^{q_m} dx \right)^{1/q_m} \\ \le & \ c \prod_{i=1}^n \left(\int_\Omega w_i |D_i\phi|^{p_i} dx \right)^{1/np_i} \le c \left(\frac{\|M\|_{L^r(\Omega)}}{\nu} |\{u^\beta > t\}|^{1-\frac{1}{r}} \right)^{\frac{1}{\bar{p}}}. \end{aligned} \tag{2.6}$$

For $T > t$ we have

$$\begin{aligned} (T-t)^{q_m} |\{u^\beta > T\}| = & \int_{\{u^\beta > T\}} (T-t)^{q_m} dx \\ \le & \int_{\{u^\beta > T\}} (u^\beta - t)^{q_m} dx \le \int_{\{u^\beta > t\}} (u^\beta - t)^{q_m} dx. \end{aligned} \tag{2.7}$$

From (2.6) and (2.7) we get

$$|\{u^\beta > T\}| \le c^{q_m} \left(\frac{\|M\|_{L^r(\Omega)}}{\nu} \right)^{\frac{q_m}{\bar{p}}} \frac{1}{(T-t)^{q_m}} |\{u^\beta > t\}|^{(1-\frac{1}{r})\frac{q_m}{\bar{p}}}$$

for every T, t with $T > t \ge t_0$. We set $\chi(t) = |\{u^\beta > t\}|$, $\tilde{C} = c^{q_m} \left(\frac{\|M\|_{L^r(\Omega)}}{\mu} \right)^{\frac{q_m}{\bar{p}}}$, $a = q_m$ and $b = (1 - \frac{1}{r})\frac{q_m}{\bar{p}}$. We use Lemma 2.2 and we get

$$|\{u^\beta > t_0 + c_*\}| = 0,$$

that is

$$u^\beta \le t_0 + c_*$$

almost everywhere in Ω, where

$$c_* = c \left(\frac{\|M\|_{L^r(\Omega)}}{\mu} \right)^{\frac{1}{\bar{p}}} |\Omega|^{\left[(1-\frac{1}{r})\frac{q_m}{\bar{p}} - 1\right]\frac{1}{q_m}} 2^{(1-\frac{1}{r})\frac{q_m}{\bar{p}}\left[(1-\frac{1}{r})\frac{q_m}{\bar{p}} - 1\right]^{-1}}.$$

In order to get the right hand side of (1.4), we take a sequence $\{(t_0)_m\}_m$ with $(t_0)_m \to \sup_{\partial\Omega} u^\beta$. We apply the right hand side of (1.4) to $-u$ and we get the left hand side of (1.4). This ends the proof of Theorem 1.

Proof of Theorem 1.3. We assume that $\tilde{A}, A \in \mathrm{R}^{N\times n}$ with $\tilde{A}^\beta = 0$ and $\tilde{A}^\alpha = A^\alpha$ for every $\alpha \neq \beta$. We need to show that

$$\nu \sum_{i=1}^{n} w_i |A_i - \tilde{A}_i|^{p_i} \leq \sum_{i=1}^{n} \sum_{\alpha=1}^{N} t_i w_i |A_i|^{p_i-2} A_i^\alpha (A_i^\alpha - \tilde{A}_i^\alpha), \tag{2.6}$$

holds true with $\nu = \min\limits_{1\leq i\leq n} \{t_i\}$. In fact

$$\begin{aligned}
& \sum_{i=1}^{n} \sum_{\alpha=1}^{N} t_i w_i |A_i|^{p_i-2} A_i^\alpha (A_i^\alpha - \tilde{A}_i^\alpha) \\
= & \sum_{i=1}^{n} t_i w_i |A_i|^{p_i-2} (A_i^\beta)^2 \geq \min_{1\leq i\leq n} \{t_i\} \sum_{i=1}^{n} w_i |A_i^\beta|^{p_i-2} (A_i^\beta)^2 \\
= & \min_{1\leq i\leq n} \{t_i\} \sum_{i=1}^{n} w_i |A_i^\beta|^{p_i} = \min_{1\leq i\leq n} \{t_i\} \sum_{i=1}^{n} w_i |A_i - \tilde{A}_i|^{p_i}.
\end{aligned}$$

Using (2.6) we derive

$$\begin{aligned}
& \sum_{i=1}^{n} \sum_{\alpha=1}^{N} a_i^\alpha(x, A)(A_i^\alpha - \tilde{A}_i^\alpha) \\
= & \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \left[t_i w_i |A_i|^{p_i-2} A_i^\alpha + \frac{m(x)}{1 + \|A\|} \right] (A_i^\alpha - \tilde{A}_i^\alpha) \\
\geq & \ \nu \sum_{i=1}^{n} w_i |A_i - \tilde{A}_i|^{p_i} + \frac{m(x)}{1 + \|A\|} \sum_{i=1}^{n} \sum_{\alpha=1}^{N} (A_i^\alpha - \tilde{A}_i^\alpha) \\
= & \ \nu \sum_{i=1}^{n} w_i |A_i - \tilde{A}_i|^{p_i} + \frac{m(x)}{1 + \|A\|} \sum_{i=1}^{n} A_i^\beta \\
\geq & \ \nu \sum_{i=1}^{n} w_i |A_i - \tilde{A}_i|^{p_i} - \frac{m(x)}{1 + \|A\|} \sum_{i=1}^{n} |A_i^\beta|.
\end{aligned} \tag{2.7}$$

Using the inequality between geometric and arithmetic means

$$\frac{1}{n} \sum_{i=1}^{n} |w_i| \leq \left(\frac{1}{n} \sum_{i=1}^{n} |w_i|^2 \right)^{1/2},$$

and recall that $M(x) = n^{1/2}m(x)$, we get from (2.7) that

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{\alpha=1}^{N} a_i^\alpha(x,A)(A_i^\alpha - \tilde{A}_i^\alpha) + M(x) \\
\geq\ & \nu\sum_{i=1}^{n} w_i|A_i - \tilde{A}_i|^{p_i} - \frac{m(x)}{1+\|A\|}\sum_{i=1}^{n}|A_i^\beta| + M(x) \\
\geq\ & \nu\sum_{i=1}^{n} w_i|A_i - \tilde{A}_i|^{p_i} - m(x) + M(x) \\
\geq\ & \nu\sum_{i=1}^{n} w_i|A_i - \tilde{A}_i|^{p_i}.
\end{aligned}$$

Thus the monotonicity inequality (1.2) holds true with $\nu = \min_{1\leq i\leq n}\{t_i\}$ and $M(x) = m(x)$.

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