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The classification of 2-dimensional Jordan-ω-Lie algebras

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Abstract

A complex Jordan-ω-Lie algebra is a vector space over the complex field, equipped with a symmetric bracket $[-,-]$ and a bilinear form ω such that

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=\omega(x,y)z+\omega(y,z)x+\omega(z,x)y$$

for all $x,y,z\in L$. In this paper, we provide an approach to classify all 2-dimensional non-Jordan-Lie complex Jordan-ω-Lie algebras.

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1 Introduction

We define a Jordan-ω-Lie algebra over a field of characteristic zero.

Let K be a field of characteristic zero and L be a finite dimensional vector space over K. Let $[-,-]:L\times L\longmapsto L$ be a symmetric bracket on L and $\omega:L\times L\longmapsto K$ be a bilinear form on L. The triple $(L,\ [-,-],\ \omega)$ is called a Jordan-ω-Lie algebra, if it satisfies

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=\omega(x,y)z+\omega(y,z)x+\omega(z,x)y, \tag{1.1}$$

for all $x,y,z\in L$.

The equation (1.1) is called the ω-Jacobi identity. Apparently, a Jordan-ω-Lie algebra is a Jordan-Lie algebra([3]) if and only if $\omega\equiv 0$. We call the Jordan-Lie algebras trivial Jordan-ω-Lie algebras.

If $[-,-]$ is skew-symmetric ,The triple $(L,\ [-,-],\ \omega)$ is called a ω-Lie algebra. ω-Lie algebras was introduced in the work of Nurowski [2]. A fundamental development of ω-Lie algebras was by Zusmanovich [4]. We usually call the Lie algebras trivial ω-Lie algebras. It's easy to see that all ω-Lie algebras are trivial in the case of dimension 1 and 2. However, 2-dimensional Jordan-ω-Lie algebras are not all trivial. The purpose of this paper is to give a classification of all 2-dimensional nontrivial Jordan-ω-Lie algebras over the field of complex numbers.

2 Preliminary Notes

Let L be a Jordan-ω-Lie algebra with a basis $\{x_1, x_2, \cdots, x_n\}$. The metric matrix of ω can be found, i.e.

$$\begin{pmatrix} \omega(x_1,x_1) & \omega(x_1,x_2) & \cdots & \omega(x_1,x_n) \\ \omega(x_2,x_1) & \omega(x_2,x_2) & \cdots & \omega(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(x_n,x_1) & \omega(x_n,x_2) & \cdots & \omega(x_n,x_n) \end{pmatrix}.$$

In [1] and [2], the classification of 3-dimensional nontrivial ω-Lie algebras has been gave by considering the dimension of $[L, L]$ case by case. In the note, we will discuss the dimension of $[L, L]$, which maybe 0, 1 and 2. In what follows, $\mathbb{C}$ is the field of complex numbers and L denotes a Jordan-ω-Lie algebra with a basis of $\{x, y\}$. All the coefficients of x and y appeared are in $\mathbb{C}$. $[L, L]$ is called the commutator subalgebra of L and we use $L^{(1)}$ to denote it.

Part 1 $DimL^{(1)} = 0$ or 1

If $dimL^{(1)} = 0$, it's easy to see that L is trivial.

If $dimL^{(1)} = 1$, we choose x as a basis of $L^{(1)}$ and $y \notin L^{(1)}$. Suppose that

$$\begin{cases} [x,x] = k_1 x \\ [x,y] = k_2 x \ . \\ [y,y] = k_3 x \end{cases} \tag{2.1}$$

One of k_1, k_2 and k_3 is not zero.

There are four fundamental ω-Jacobi identity on L, i.e.

$$3[[x,x],x] = 3\omega(x,x)x, \tag{2.2}$$

$$3[[y,y],y] = 3\omega(y,y)y, \tag{2.3}$$

$$[[x,x],y] + 2[[x,y],x] = \omega(x,x)y + (\omega(x,y) + \omega(y,x))x, \tag{2.4}$$

$$[[y,y],x] + 2[[x,y],y] = \omega(y,y)x + (\omega(x,y) + \omega(y,x))y. \tag{2.5}$$

According to the equation set (2.1), the equations (2.2), (2.3), (2.4), (2.5) are equal to

$$k_1{}^2x = \omega(x,x)x, \tag{2.6}$$

$$k_2k_3x = \omega(y,y)y, \tag{2.7}$$

$$3k_1k_2x = \omega(x,x)y + (\omega(x,y)+\omega(y,x))x, \tag{2.8}$$

$$(k_1k_3+2k_2{}^2)x = \omega(y,y)x + (\omega(x,y)+\omega(y,x))y. \tag{2.9}$$

Comparing the coefficients of x and y in the two-hand sides in the equations (2.6), (2.7), (2.8), (2.9) and summing up the equation sets, we obtain

$$\begin{cases} k_1 = 0 \\ k_2 = 0 \\ k_3 \neq 0 \\ \omega(x,x) = 0 \\ \omega(y,y) = 0 \\ \omega(x,y)+\omega(y,x) = 0 \end{cases}. \tag{2.10}$$

In the case, let $x' = k_3x$. Then we obtain

$$[x',x'] = 0, \quad [x',y] = 0, \quad [y,y] = x',$$

$$\omega(x',x') = 0, \quad \omega(y,y) = 0, \quad \omega(x',y)+\omega(y,x') = 0.$$

We get a 2-dimensional Jordan-ω-Lie algebra:

$$L_2{}^1 : [x,x] = 0, \quad [x,y] = 0, \quad [y,y] = x, \quad A = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} (\alpha \neq 0).$$

If $\alpha = 0$, $L_2{}^1$ is trivial.

Part 2 $DimL^{(1)} = 2$

If $dimL^{(1)} = 2$, we choose $\{x,y\}$ as of a basis of both L and $L^{(1)}$. Assume that $[x,x] = k_1x + l_1y$. Our arguments will be separated into the following two cases: $l_1 = 0$ or $l_1 \neq 0$.

Case 1. If $l_1 = 0$, $[x,x] = k_1x$. Moreover, If $k_1 = 0$, $[x,x] = 0$. If $k_1 \neq 0$, let $x' = \frac{1}{k_1}x$ and then $[x',x'] = x'$. Hence this case can be separated into two subcases: $[x,x] = 0$ and $[x,x] = x$.

Subcase 1. We suppose that

$$\begin{cases} [x,x] = 0 \\ [x,y] = k_2x + l_2y \\ [y,y] = k_3x + l_3y \end{cases}. \tag{2.11}$$

Because $dimL^{(1)} = 2$, one of k_2 and k_3 is not zero as well as one of l_2 and l_3 is not zero.

We will follow the idea in the case of $dimL^{(1)} = 1$. According to the the equations (2.2), (2.3), (2.4), (2.5) and the equation set (2.11), we obtain

$$\begin{cases} l_2 = 0 \\ k_3(k_2 + l_3) = 0 \\ k_2 l_3 + 2{k_2}^2 = {l_3}^2 \\ \omega(x,x) = 0 \\ \omega(y,y) = {l_3}^2 \\ \omega(x,y) + \omega(y,x) = 0 \end{cases} . \tag{2.12}$$

We have $l_3 \neq 0$ because of $l_2 = 0$. The third equation of (2.12) can be divided ${l_3}^2$. Then

$$\begin{cases} l_2 = 0 \\ k_3 = 0 \quad or \quad k_2 = -l_3 \\ \frac{k_2}{l_3} + 2(\frac{k_2}{l_3})^2 = 1 \end{cases} . \tag{2.13}$$

The equation set (2.13) can be separated into the following two situations:

$$\begin{cases} l_2 = 0 \\ k_3 = 0 \\ k_2 = \frac{1}{2} l_3 \end{cases}, \qquad or \qquad \begin{cases} l_2 = 0 \\ k_3 \in \mathbb{C} \\ k_2 = -l_3 \end{cases} .$$

In the first situation, let $y' = \frac{1}{l_3} y$. Then we get a Jordan-ω-Lie algebra:

$${L_2}^2: \quad [x,x] = 0, \quad [x,y] = \frac{1}{2}x, \quad [y,y] = y, \quad A = \begin{pmatrix} 0 & \alpha \\ -\alpha & 1 \end{pmatrix}.$$

In the second situation, if $k_3 \neq 0$, let $x' = \frac{k_3}{{l_3}^2} x$ and $y' = -\frac{1}{l_3} x$. Then there is a Jordan-ω-Lie algebra:

$${L_2}^3: \quad [x,x] = 0, \quad [x,y] = x, \quad [y,y] = x - y, \quad A = \begin{pmatrix} 0 & \alpha \\ -\alpha & 1 \end{pmatrix}.$$

If $k_3 = 0$, let $y' = \frac{1}{3}x - \frac{1}{l_3} y$. We will get the same result of the case of $k_3 \neq 0$.

Subcases 2. Suppose that $[x,x] = x$ and $[x,y] = k_2 x + l_2 y$. If $k_2 = 0$, we suppose that

$$\begin{cases} [x,x] = x \\ [x,y] = l_2 y \\ [y,y] = k_3 x + l_3 y \end{cases} . \tag{2.14}$$

If $k_2 \neq 0$, let $y' = \frac{1}{k_2} y$. Then $[x,y'] = x + l_2 y'$. We can suppose that

$$\begin{cases} [x,x] = x \\ [x,y] = x + l_2 y \\ [y,y] = k_3 x + l_3 y \end{cases} . \tag{2.15}$$

We will follow the idea in the case of $dimL^{(1)} = 1$. According to the the equations (2.2), (2.3), (2.4), (2.5) and the equation set (2.14), obtain

$$\begin{cases} l_2 = -1 \quad or \quad l_2 = \frac{1}{2} \\ k_3 = 0 \quad or \quad l_3 = 0 \\ k_3 + 2k_3 l_2 = k_3 l_2 + {l_3}^2 \\ \omega(x,x) = 1 \\ \omega(y,y) = k_3 l_2 + {l_3}^2 \\ \omega(x,y) + \omega(y,x) = 0 \end{cases}. \tag{2.16}$$

The equation set (2.16) can be separated into the following three situations:

$$\begin{cases} l_2 = -1 \\ k_3 = 0 \\ l_3 = 0 \end{cases}, \qquad \begin{cases} l_2 = -1 \\ l_3 = 0 \\ k_3 \neq 0 \end{cases} \qquad or \qquad \begin{cases} l_2 = \frac{1}{2} \\ k_3 = 0 \\ l_3 = 0 \end{cases}.$$

In the first situation,we transpose x and y and will get the same results of ${L_2}^3$.

In the second situation, let $y' = \frac{1}{\sqrt{l_3}} y$. Then we obtain a new Jordan-ω-Lie algebra:

$${L_2}^4 : [x,x] = x, \quad [x,y] = -y, \quad [y,y] = x, \quad A = \begin{pmatrix} 1 & \alpha \\ -\alpha & -1 \end{pmatrix}.$$

In the third situation, we transpose x and y and will get the same results of ${L_2}^2$.

Follow the idea in the case of $dimL^{(1)} = 1$ again. According to the the equations (2.2), (2.3), (2.4), (2.5) and the equation set (2.15), we obtain

$$\begin{cases} l_2 = -1 \quad or \quad l_2 = \frac{1}{2} \\ k_3 = 0 \quad or \quad l_3 = -1 \\ k_3 + l_3 + 2 + 2k_3 l_2 = k_3 l_2 + {l_3}^2 \\ 3l_2 l_3 + 2l_2 = 3 + 2l_2 \\ \omega(x,x) = 1 \\ \omega(y,y) = k_3 l_2 + {l_3}^2 \\ \omega(x,y) + \omega(y,x) = 3 + 2l_2 \end{cases}, \tag{2.17}$$

The equation set (2.17) can be separated into the following two situations:

$$\begin{cases} l_2 = -1 \\ l_3 = -1 \\ k_3 \in \mathbb{C} \end{cases} \qquad or \qquad \begin{cases} l_2 = \frac{1}{2} \\ k_3 = 0 \\ l_3 = 2 \end{cases}.$$

In the first situation, let $k_3 = \mu$. We obtain a family of Jordan-ω-Lie algebras of one parameter:

$$(L_2{}^5)_\mu : [x,x] = x, \quad [x,y] = x-y, \quad [y,y] = \mu x - y, \quad A = \begin{pmatrix} 1 & \alpha \\ 1-\alpha & -\mu+1 \end{pmatrix}..$$

In the second situation, let $y' = \frac{1}{2}y$. Therefore, we get a new Jordan-ω-Lie algebra

$$L_2{}^6 : [x,x] = x, \quad [x,y] = \frac{1}{2}x - y, \quad [y,y] = y, \quad A = \begin{pmatrix} 1 & \alpha \\ \frac{1}{2}-\alpha & 1 \end{pmatrix}.$$

Case 2. If $l_3 \neq 0$, $[x,x] = k_3 + l_3 y$. Let $y' = [x,x]$. Apparently, x and y' are linearly independent. We can choose $\{x, y'\}$ as a basis of L and $L^{(1)}$. Suppose that

$$\begin{cases} [x,x] = y \\ [x,y] = k_2 x + l_2 y \\ [y,y] = k_3 x + l_3 y \end{cases}. \tag{2.18}$$

One of k_2 and k_3 is not zero.

Follow the idea in the case of $dimL^{(1)} = 1$. According to the the equations (2.2), (2.3), (2.4), (2.5) and the equation set (2.18), we obtain

$$\begin{cases} l_2 = 0 \\ l_3 = -k_2 \\ \omega(x,x) = k_2 \\ \omega(y,y) = k_2{}^2 \\ \omega(x,y) + \omega(y,x) = k_3 \end{cases}. \tag{2.19}$$

Let $k_2 = \nu$ and $k_3 = \rho$, and we obtain a family of Jordan-ω-Lie algebras of two parameters:

$$(L_2{}^7)_{\nu,\rho} : [x,x] = y, \quad [x,y] = \nu x, \quad [y,y] = \rho x - \nu y, \quad A = \begin{pmatrix} \nu & \alpha \\ \rho-\alpha & \nu^2 \end{pmatrix}.$$

One of ν and ρ is not zero. If ν and ρ are all zero, transpose x and y, and we will get the same result as the case of $dimL^{(1)} = 1$.

3 Main Results

According to the preliminary notes, we obtain the main result of the paper.

Theorem 3.1. *Let L be a nontrivial 2-dimensional complex Jordan-ω-Lie algebra. $\{x, y\}$ is a basis of L. We wrte $\omega(x,y) = \alpha(\alpha \in \mathbb{C})$ and record the* *metric matrix of ω as A. Then L must be isomorphic to one of the following seven Jordan-ω-Lie algebras:*

(1) ${L_2}^1$: $[x,x]=0,\ [x,y]=0,\ [y,y]=x,\ A=\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} (\alpha \neq 0)$.

(2) ${L_2}^2$:$[x,x]=0,\ [x,y]=\frac{1}{2}x,\ [y,y]=y,\ A=\begin{pmatrix} 0 & \alpha \\ -\alpha & 1 \end{pmatrix}$.

(3) ${L_2}^3$: $[x,x]=0,\ [x,y]=x,\ [y,y]=x-y,\ A=\begin{pmatrix} 0 & \alpha \\ -\alpha & 1 \end{pmatrix}$.

(4) ${L_2}^4$: $[x,x]=x,\ [x,y]=-y,\ [y,y]=x,\ A=\begin{pmatrix} 1 & \alpha \\ -\alpha & -1 \end{pmatrix}$.

(5) $({L_2}^5)_\mu$:$[x,x]=x,\ [x,y]=x-y,\ [y,y]=\mu x-y,\ A=\begin{pmatrix} 1 & \alpha \\ 1-\alpha & -\mu+1 \end{pmatrix}$.

(6) ${L_2}^6$: $[x,x]=x,\ [x,y]=\frac{1}{2}x-y,\ [y,y]=y,\ A=\begin{pmatrix} 1 & \alpha \\ \frac{1}{2}-\alpha & 1 \end{pmatrix}$.

(7) $({L_2}^7)_{\nu,\rho}$: $[x,x]=y,\ [x,y]=\nu x,\ [y,y]=\rho x-\nu y,\ A=\begin{pmatrix} \nu & \alpha \\ \rho-\alpha & \nu^2 \end{pmatrix}$.

One of ν and ρ are not zero.

Remark 1. There is an example of ${L_2}^1$. Let L be a vector space with a basis $\{A,B\}$ over $\mathbb{C}$.

$$A=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad and \qquad B=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

A symmetric bracket on L $[-,-]$ satisfies $[x,y]=\frac{1}{2}(xy+yx)$, for all $x,y\in L$. A bilinear form on L ω satisfies $\omega(A,A)=0$, $\omega(B,B)=0$ and $\omega(A,B)=-\omega(B,A)(\omega(A,B)\neq 0)$. The triple $(L,\ [-,-],\ \omega)$ satisfies all the ω-Jacobi identities on L and is a Jordan-ω-Lie algebra.

Remark 2. Let L be a complex Jordan-ω-Lie algebra. If $dimL=2$, ω may not be symmetric according to theorem 3.1. If $dimL\geq 3$, ω must be symmetric.

For any linearly independent $x,y,z\in L$, write the ω-Jacobi identity (1.1) for the triple x, y, z. Transpose x and y and compare the two-hand sides of the two equations. Because $[-,-]$ is symmetric, the left-hand side of the two equations are equal. According to the right-hand sides of the two equations, we obtain $\omega(x,y)=\omega(y,x)$. Hence ω is symmetric because x and y are arbitrarily.

Remark 3. There are 3-dimensional Jordan-ω-Lie algebras.

Let L be a vector space with a basis $\{x,y,z\}$ over $\mathbb{C}$. A symmetric bracket on L $[-,-]$ $[-,-]$ satisfies

$$[x,x]=x,\quad [y,y]=y,\quad [z,z]=z$$

$$[x,y]=\frac{1}{2}x+\frac{1}{2}y,\quad [x,z]=\frac{1}{2}x+\frac{1}{2}z,\quad [y,z]=\frac{1}{2}y+\frac{1}{2}z.$$

A bilinear form on L ω satisfies

$$\omega(x,x) = \omega(y,y) = \omega(z,z) = \omega(x,y) = \omega(x,z) = \omega(y,z) = 1.$$

The triple $(L,\ [-,-],\ \omega)$ satisfies all the ω-Jacobi identities on L and is a Jordan-ω-Lie algebra.

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