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Bialgebra structures on 3-Lie algebra L_d

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Abstract

In this paper, we discuss bialgebra structures on the 3-Lie algebra L_d. It is proved that there do not exist 3-Lie bialgebras of types (L_d, C_{b_2}), (L_d, C_{c_1}), (L_d, C_{c_2}), (L_d, C_d) and (L_d, C_e), and there exist only five classes of 3-Lie bialgebras of types on L_d which are $(L_d, 0)$, (L_d, C_{b_1}, Δ_1), (L_d, C_{b_1}, Δ_2), (L_d, C_{c_3}, Δ_1) and (L_d, C_{c_3}, Δ_2).

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1 Preliminaries

In papers [1, 2, 3], the 3-Lie bialgebras of types (L_b, C_b), (L_b, C_c), (L_b, C_d),(L_b, C_e), (L_c, C_b), (L_c, C_c), (L_c, C_d) and (L_c, C_e) are studied. In this paper we discuss the structures of 3-Lie bialgebras of types (L_d, C_b), (L_d, C_c), (L_d, C_d) and (L_d, C_e) over the field of complex numbers.

We know that for a vector space over a field F, and a linear map $\Delta : V \to V \otimes V \otimes V$, (V, Δ) is a 3-Lie coalgebra if and only if (V^*, Δ^*) is a 3-Lie algebra, where for all $f \in V^*, x_1, x_2, x_3 \in V$, $\langle \Delta(f), x_1 \otimes x_2 \otimes x_3 \rangle = \langle f, \Delta^*(x_1, x_2, x_3) \rangle$. So we can obtain the classification of m-dimensional 3-Lie coalgebras from the classification of m-dimensional 3-Lie algebras. In this paper, suppose that F is a field of all the complex numbers. And in the multiplication of 3-Lie algebras (3-Lie coalgebras), we may omit the zero product of basis vectors.

First, we give some notions.

A 3-Lie bialgebra[4] is a triple (L, μ, Δ) such that

(1) (L, μ) is a 3-Lie algebra with the multiplication $\mu : L \wedge L \wedge L \to L$,

(2) (L, Δ) is a 3-Lie coalgebra with $\Delta : L \to L \wedge L \wedge L$,
(3) Δ and μ satisfy the following identity, for $x, y, u, v, w \in L$,

$\Delta\mu(x, y, z) = ad_\mu^{(3)}(x, y)\Delta(z) + ad_\mu^{(3)}(y, z)\Delta(x) + ad_\mu^{(3)}(z, x)\Delta(y)$,

where $ad_\mu^{(3)}(x, y)$, $ad_\mu^{(3)}(z, x)$, $ad_\mu^{(3)}(y, z) : L \otimes L \otimes L \to L \otimes L \otimes L$ are linear maps defined by (similar for $ad_\mu^{(3)}(z, x)$ and $ad_\mu^{(3)}(y, z)$)

$ad_\mu^{(3)}(x, y)(u \otimes v \otimes w) = (ad_\mu(x, y) \otimes 1 \otimes 1)(u \otimes v \otimes w)$
$+(1 \otimes ad_\mu(x, y) \otimes 1)(u \otimes v \otimes w) + (1 \otimes 1 \otimes ad_\mu(x, y))(u \otimes v \otimes w)$
$= \mu(x, y, u) \otimes v \otimes w + u \otimes \mu(x, y, v) \otimes w + u \otimes v \otimes \mu(x, y, w)$.

Lemma 2.1[5] *Let (L, μ) be a 4-dimensional 3-Lie algebra with a basis e_1, e_2, e_3, e_4. Then L is isomorphic to one and only one of the following: L_a is abelian.*

$L_{b_1}.\mu(e_2, e_3, e_4) = e_1$. $L_{b_2}.\mu(e_1, e_2, e_3) = e_1$.
$L_{c_1}.\mu_{c_1}(e_2, e_3, e_4) = e_1, \quad \mu_{c_1}(e_1, e_3, e_4) = e_2$.
$L_{c_2}.\mu_{c_2}(e_2, e_3, e_4) = \alpha e_1 + e_2, \quad \mu_{c_2}(e_1, e_3, e_4) = e_2,\ \alpha \in F, \alpha \neq 0$.
$L_{c_3}.\mu_{c_3}(e_1, e_3, e_4) = e_1, \mu_{c_3}(e_2, e_3, e_4) = e_2$.
$L_d.\ \mu_d(e_2, e_3, e_4) = e_1, \mu_d(e_1, e_3, e_4) = e_2, \mu_d(e_1, e_2, e_4) = e_3$.
$L_e.\ \mu_e(e_2, e_3, e_4) = e_1, \mu_e(e_1, e_3, e_4) = e_2, \mu_e(e_1, e_2, e_4) = e_3, \mu_e(e_1, e_2, e_3) = e_4$.

Lemma 2.2[6] *Let L be a vector space over a field F, $\Delta : L \to L \otimes L \otimes L$ be a linear mapping. Then (L, Δ) is a 3-Lie coalgebra if and only if (L^*, Δ^*) is a 3-Lie algebra, where L^* is the dual space of L, and Δ^* is the dual mapping of Δ.*

For convenience, in the following, for a 3-Lie bialgebra (L, μ, Δ), if the 3-Lie algebra (L, μ) is the case (L, μ_d) in Lemma 2.1 and the 3-Lie coalgebra (L, Δ) is the case (L, Δ_{c_1}) for example, then the 3-Lie bialgebra (L, μ_d, Δ_{c_1}) is simply denoted by (L_d, C_{c_1}), which is called *the 3-Lie bialgebra of type* (L_d, C_{c_1}).

2 Bialgebra structures on 3-Lie algebra L_d

For a given 3-Lie algebra L, in order to find all the 3-Lie bialgebra structures on L, we should find all the 3-Lie coalgebra structures on L which are compatible with the 3-Lie algebra L. Although a permutation of a basis of L gives isomorphic 3-Lie coalgebra, but it may lead to the non-equivalent 3-Lie bialgebra.

Theorem 3.1 The non-equivalent 3-Lie bialgebras of the type (L_d, C_{c_3}) are as follows: (L_d, C_{c_3}, Δ_1). $\Delta_1(e_1) = e_1 \wedge e_3 \wedge e_4, \Delta_1(e_2) = e_2 \wedge e_3 \wedge e_4$;
(L_d, C_{c_3}, Δ_2). $\Delta_2(e_1) = e_1 \wedge e_2 \wedge e_4, \Delta_2(e_3) = e_3 \wedge e_2 \wedge e_4$;
(L_d, C_{c_3}, Δ_3). $\Delta_3(e_1) = e_1 \wedge e_3 \wedge e_2, \Delta_3(e_4) = e_4 \wedge e_3 \wedge e_2$;
(L_d, C_{c_3}, Δ_4). $\Delta_4(e_2) = e_2 \wedge e_3 \wedge e_1, \Delta_4(e_4) = e_4 \wedge e_3 \wedge e_1$;
(L_d, C_{c_3}, Δ_5). $\Delta_5(e_3) = e_3 \wedge e_1 \wedge e_2, \Delta_5(e_4) = e_4 \wedge e_1 \wedge e_2$.

Proof We need to verify that wether the following twelve isomorphic 3-Lie coalgebras of the type C_{c_3} are compatible with the 3-Lie algebra L_d, respectively:
(1).$\Delta(e_1) = e_1 \wedge e_3 \wedge e_4, \Delta(e_2) = e_2 \wedge e_3 \wedge e_4$; (2).$\Delta(e_1) = e_1 \wedge e_4 \wedge e_3$, $\Delta(e_2) = e_2 \wedge e_4 \wedge e_3$; (3).$\Delta(e_1) = e_1 \wedge e_2 \wedge e_4, \Delta(e_3) = e_3 \wedge e_2 \wedge e_4$;
(4).$\Delta(e_1) = e_1 \wedge e_4 \wedge e_2, \Delta(e_3) = e_3 \wedge e_4 \wedge e_2$; (5).$\Delta(e_1) = e_1 \wedge e_3 \wedge e_2$, $\Delta(e_4) = e_4 \wedge e_3 \wedge e_2$; (6).$\Delta(e_1) = e_1 \wedge e_2 \wedge e_3, \Delta(e_4) = e_4 \wedge e_2 \wedge e_3$;
(7).$\Delta(e_2) = e_2 \wedge e_4 \wedge e_1, \Delta(e_3) = e_3 \wedge e_4 \wedge e_1$; (8).$\Delta(e_2) = e_2 \wedge e_1 \wedge e_4$,

$\Delta(e_3) = e_3 \wedge e_1 \wedge e_4$; (9).$\Delta(e_2) = e_2 \wedge e_3 \wedge e_1, \Delta(e_4) = e_4 \wedge e_3 \wedge e_1$;
(10).$\Delta(e_2) = e_2 \wedge e_1 \wedge e_3, \Delta(e_4) = e_4 \wedge e_1 \wedge e_3$; (11).$\Delta(e_3) = e_3 \wedge e_1 \wedge e_2$,
$\Delta(e_4) = e_4 \wedge e_1 \wedge e_2$; (12).$\Delta(e_3) = e_3 \wedge e_2 \wedge e_1, \Delta(e_4) = e_4 \wedge e_2 \wedge e_1$.

By a direct computation, the twelve 3-Lie coalgebras are compatible with the 3-Lie algebra L_d. By the following isomorphisms of the 4-Lie bialgebras
$(1) \to (2), (11) \to (12): \ f(e_1) = e_1, f(e_2) = -e_2, f(e_3) = -e_3, f(e_4) = e_4$;
$(1) \to (8),\ (2) \to (7): \ f(e_1) = -e_3, f(e_2) = e_2, f(e_3) = e_1, f(e_4) = e_4$;
$(3) \to (4), (5) \to (6), (9) \to (10): f(e_1) = -e_1, f(e_2) = -e_2, f(e_3) = e_3, f(e_4) = e_4$;
we get 3-Lie bialgebras (L_d, C_{c_3}, Δ_i), $1 \leq i \leq 5$. If h is an automorphism of 3-Lie algebra L_d satisfying that $\Delta_2 h(e_j) = h(\Delta_1(e_j))$ for $j = 1, 2, 3, 4$, then we have $h(e_1) = 0$. Contradiction. Therefore, (L_d, C_{c_3}, Δ_1) and (L_d, C_{c_3}, Δ_2) are non-equivalent. By the similar discussion and the derived algebra of 3-Lie algebras $L_d^1 = Fe_1 + Fe_2 + Fe_3$, we obtain (L_d, C_{c_3}, Δ_j), $j = 1, 2, 3, 4, 5$ are non-equivalent.

Theorem 3.2 The non-equivalent 3-Lie bialgebras of the type (L_d, C_{b_1}) are as follows:

(L_d, C_{b_1}, Δ_1): $\Delta_1(e_4) = e_1 \wedge e_2 \wedge e_3$; (L_d, C_{b_1}, Δ_2): $\Delta_2(e_4) = e_1 \wedge e_3 \wedge e_2$.

Proof By a direct computation, eight 3-Lie coalgebras of type C_{b_1}, which are obtained by permuting a basis e_1, e_2, e_3, e_4, are compatible with L_d:
(1) $\Delta\ (e_1) = e_2 \wedge e_3 \wedge e_4$; (2) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,$; (3) $\Delta(e_2) = e_1 \wedge e_4 \wedge e_3$;
(4) $\Delta(e_2) = e_1 \wedge e_3 \wedge e_4$; (5) $\Delta(e_3) = e_1 \wedge e_2 \wedge e_4$; (6) $\Delta(e_3) = e_1 \wedge e_4 \wedge e_2$;
(7) $\Delta(e_4) = e_1 \wedge e_2 \wedge e_3$; (8) $\Delta(e_4) = e_2 \wedge e_1 \wedge e_3$.
The discussion is similar to Theorem 3.1, the non-equivalent 3-Lie bialgebras of type (L_d, C_{b_1}) are only (L_d, C_{b_1}, Δ_1) and (L_d, C_{b_1}, Δ_2). We omit the discussion process.

Theorem 3.3 There do not exist 3-Lie bialgebras of types (L_d, C_{b_2}), (L_d, C_{c_1}), (L_d, C_{c_2}), (L_d, C_d) and (L_d, C_e).

Proof By a direct computation and Lemma 2.1, we have that the twenty-four isomorphic 3-Lie coalgebras of the type C_{b_2}:
(1)$\Delta(e_1) = e_1 \wedge e_2 \wedge e_3$; (2)$\Delta(e_1) = e_1 \wedge e_2 \wedge e_4$; (3)$\Delta(e_1) = e_1 \wedge e_3 \wedge e_4$;
(4)$\Delta(e_1) = e_1 \wedge e_3 \wedge e_2$; (5)$\Delta(e_1) = e_1 \wedge e_4 \wedge e_2$; (6)$\Delta(e_1) = e_1 \wedge e_4 \wedge e_3$;
(7)$\Delta(e_2) = e_2 \wedge e_1 \wedge e_3,$; (8)$\Delta(e_2) = e_2 \wedge e_1 \wedge e_4$; (9)$\Delta(e_2) = e_2 \wedge e_3 \wedge e_4$;
(10)$\Delta(e_2) = e_2 \wedge e_3 \wedge e_1$; (11)$\Delta(e_2) = e_2 \wedge e_4 \wedge e_1$; (12)$\Delta(e_2) = e_2 \wedge e_4 \wedge e_3$;
(13)$\Delta(e_3) = e_3 \wedge e_1 \wedge e_2$; (14)$\Delta(e_3) = e_3 \wedge e_1 \wedge e_4$; (15)$\Delta(e_3) = e_3 \wedge e_2 \wedge e_4$;
(16)$\Delta(e_3) = e_3 \wedge e_2 \wedge e_1$; (17)$\Delta(e_3) = e_3 \wedge e_4 \wedge e_1$; (18)$\Delta(e_3) = e_3 \wedge e_4 \wedge e_2$;
(19)$\Delta(e_4) = e_4 \wedge e_1 \wedge e_2$; (20)$\Delta(e_4) = e_4 \wedge e_1 \wedge e_3$; (21)$\Delta(e_4) = e_4 \wedge e_2 \wedge e_3$;
(22)$\Delta(e_4) = e_4 \wedge e_2 \wedge e_1$; (23)$\Delta(e_4) = e_4 \wedge e_3 \wedge e_1$; (24)$\Delta(e_4) = e_4 \wedge e_3 \wedge e_2$;
the twelve isomorphic 3-Lie coalgebras of the type C_{c_1}:
(1) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_1 \wedge e_3 \wedge e_4$; (2) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3$,
$\Delta(e_2) = e_1 \wedge e_4 \wedge e_3$; (3) $\Delta(e_1) = e_3 \wedge e_2 \wedge e_4, \Delta(e_3) = e_1 \wedge e_2 \wedge e_4$;
(4) $\Delta(e_1) = e_3 \wedge e_4 \wedge e_2, \Delta(e_3) = e_1 \wedge e_4 \wedge e_2$; (5) $\Delta(e_1) = e_4 \wedge e_3 \wedge e_2$,
$\Delta(e_4) = e_1 \wedge e_3 \wedge e_2$; (6) $\Delta(e_1) = e_4 \wedge e_2 \wedge e_3, \Delta(e_4) = e_1 \wedge e_2 \wedge e_3$;
(7) $\Delta(e_2) = e_3 \wedge e_4 \wedge e_1, \Delta(e_3) = e_2 \wedge e_4 \wedge e_1$; (8) $\Delta(e_2) = e_3 \wedge e_1 \wedge e_4$,
$\Delta(e_3) = e_2 \wedge e_1 \wedge e_4$; (9) $\Delta(e_2) = e_4 \wedge e_3 \wedge e_1, \Delta(e_4) = e_2 \wedge e_3 \wedge e_1,$;
(10) $\Delta(e_2) = e_4 \wedge e_1 \wedge e_3, \Delta(e_4) = e_2 \wedge e_1 \wedge e_3,$; (11) $\Delta(e_3) = e_4 \wedge e_1 \wedge e_2$,
$\Delta(e_4) = e_3 \wedge e_1 \wedge e_2$; (12) $\Delta(e_3) = e_4 \wedge e_2 \wedge e_1, \Delta(e_4) = e_3 \wedge e_2 \wedge e_1$;
the twenty-four 3-Lie coalgebras of the type C_{c_2}:

(1) $\Delta(e_1) = \alpha e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_2 \wedge e_3 \wedge e_4 + e_1 \wedge e_3 \wedge e_4;$
(2) $\Delta(e_1) = \alpha e_2 \wedge e_4 \wedge e_3, \Delta(e_2) = e_2 \wedge e_4 \wedge e_3 + e_1 \wedge e_4 \wedge e_3;$
(3) $\Delta(e_1) = e_1 \wedge e_3 \wedge e_4 + e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = \alpha e_1 \wedge e_3 \wedge e_4;$
(4) $\Delta(e_1) = e_1 \wedge e_4 \wedge e_3 + e_2 \wedge e_4 \wedge e_3, \Delta(e_2) = \alpha e_1 \wedge e_4 \wedge e_3;$
(5) $\Delta(e_1) = e_1 \wedge e_2 \wedge e_4 + e_3 \wedge e_2 \wedge e_4, \Delta(e_3) = \alpha e_1 \wedge e_2 \wedge e_4;$
(6) $\Delta(e_1) = e_1 \wedge e_4 \wedge e_2 + e_3 \wedge e_4 \wedge e_2, \Delta(e_3) = \alpha e_1 \wedge e_4 \wedge e_2;$
(7) $\Delta(e_1) = \alpha e_3 \wedge e_4 \wedge e_2, \Delta(e_3) = e_3 \wedge e_4 \wedge e_2 + e_1 \wedge e_4 \wedge e_2;$
(8) $\Delta(e_1) = \alpha e_3 \wedge e_2 \wedge e_4, \Delta(e_3) = e_3 \wedge e_2 \wedge e_4 + e_1 \wedge e_2 \wedge e_4;$
(9) $\Delta(e_1) = e_1 \wedge e_3 \wedge e_2 + e_4 \wedge e_3 \wedge e_2, \Delta(e_4) = \alpha e_1 \wedge e_3 \wedge e_2;$
(10)$\Delta(e_1) = e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_2 \wedge e_3, \Delta(e_4) = \alpha e_1 \wedge e_2 \wedge e_3;$
(11)$\Delta(e_1) = \alpha e_4 \wedge e_2 \wedge e_3, \Delta(e_4) = e_4 \wedge e_2 \wedge e_3 + e_1 \wedge e_2 \wedge e_3;$
(12)$\Delta(e_1) = \alpha e_4 \wedge e_3 \wedge e_2, \Delta(e_4) = e_4 \wedge e_3 \wedge e_2 + e_1 \wedge e_3 \wedge e_2;$
(13)$\Delta(e_2) = \alpha e_3 \wedge e_1 \wedge e_4, \Delta(e_3) = e_3 \wedge e_1 \wedge e_4 + e_2 \wedge e_1 \wedge e_4;$
(14)$\Delta(e_2) = \alpha e_3 \wedge e_4 \wedge e_1, \Delta(e_3) = e_3 \wedge e_4 \wedge e_1 + e_2 \wedge e_4 \wedge e_1;$
(15)$\Delta(e_2) = e_2 \wedge e_4 \wedge e_1 + e_3 \wedge e_4 \wedge e_1, \Delta(e_3) = \alpha e_2 \wedge e_4 \wedge e_1;$
(16)$\Delta(e_2) = e_2 \wedge e_1 \wedge e_4 + e_3 \wedge e_1 \wedge e_4, \Delta(e_3) = \alpha e_2 \wedge e_1 \wedge e_4;$
(17)$\Delta(e_2) = \alpha e_4 \wedge e_3 \wedge e_1, \Delta(e_4) = e_4 \wedge e_3 \wedge e_1 + e_2 \wedge e_3 \wedge e_1;$
(18)$\Delta(e_2) = \alpha e_4 \wedge e_1 \wedge e_3, \Delta(e_4) = e_4 \wedge e_1 \wedge e_3 + e_2 \wedge e_1 \wedge e_3;$
(19)$\Delta(e_2) = e_2 \wedge e_1 \wedge e_3 + e_4 \wedge e_1 \wedge e_3, \Delta(e_4) = \alpha e_2 \wedge e_1 \wedge e_3;$
(20)$\Delta(e_2) = e_2 \wedge e_3 \wedge e_1 + e_4 \wedge e_3 \wedge e_1, \Delta(e_4) = \alpha e_2 \wedge e_3 \wedge e_1;$
(21)$\Delta(e_3) = \alpha e_4 \wedge e_1 \wedge e_2, \Delta(e_4) = e_4 \wedge e_1 \wedge e_2 + e_3 \wedge e_1 \wedge e_2;$
(22)$\Delta(e_3) = \alpha e_4 \wedge e_2 \wedge e_1, \Delta(e_4) = e_4 \wedge e_2 \wedge e_1 + e_3 \wedge e_2 \wedge e_1;$
(23)$\Delta(e_3) = e_3 \wedge e_2 \wedge e_1 + e_4 \wedge e_2 \wedge e_1, \Delta(e_4) = \alpha e_3 \wedge e_2 \wedge e_1;$
(24)$\Delta(e_3) = e_3 \wedge e_1 \wedge e_2 + e_4 \wedge e_1 \wedge e_2, \Delta(e_4) = \alpha e_3 \wedge e_1 \wedge e_2.$

the twenty-four isomorphic 3-Lie coalgebras of the type C_d:

(1) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_1 \wedge e_3 \wedge e_4,\ \Delta(e_3) = e_1 \wedge e_2 \wedge e_4;$

(2) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_3 \wedge e_1 \wedge e_4,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4;$

(3) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_1 \wedge e_3 \wedge e_4,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4;$

(4) $\Delta(e_1) = e_3 \wedge e_2 \wedge e_4,\ \Delta(e_2) = e_3 \wedge e_1 \wedge e_4,\ \Delta(e_3) = e_1 \wedge e_2 \wedge e_4;$

(5) $\Delta(e_1) = e_3 \wedge e_2 \wedge e_4,\ \Delta(e_2) = e_3 \wedge e_1 \wedge e_4,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4;$

(6) $\Delta(e_1) = e_3 \wedge e_2 \wedge e_4,\ \Delta(e_2) = e_1 \wedge e_3 \wedge e_4,\ \Delta(e_3) = e_1 \wedge e_2 \wedge e_4;$

(7) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_1 \wedge e_4 \wedge e_3,\ \Delta(e_4) = e_2 \wedge e_1 \wedge e_3;$

(8) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_4 \wedge e_1 \wedge e_3,\ \Delta(e_4) = e_2 \wedge e_1 \wedge e_3,$

(9) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_1 \wedge e_4 \wedge e_3,\ \Delta(e_4) = e_1 \wedge e_2 \wedge e_3;$

(10) $\Delta(e_1) = e_4 \wedge e_2 \wedge e_3,\ \Delta(e_2) = e_4 \wedge e_1 \wedge e_3,\ \Delta(e_4) = e_2 \wedge e_1 \wedge e_3;$

(11) $\Delta(e_1) = e_4 \wedge e_2 \wedge e_3,\ \Delta(e_2) = e_1 \wedge e_4 \wedge e_3,\ \Delta(e_4) = e_1 \wedge e_2 \wedge e_3;$

(12) $\Delta(e_1) = e_4 \wedge e_2 \wedge e_3,\ \Delta(e_2) = e_4 \wedge e_1 \wedge e_3,\ \Delta(e_4) = e_1 \wedge e_2 \wedge e_3;$

(13) $\Delta(e_1) = e_3 \wedge e_4 \wedge e_2,\ \Delta(e_3) = e_4 \wedge e_1 \wedge e_2,\ \Delta(e_4) = e_3 \wedge e_1 \wedge e_2;$

(14) $\Delta(e_1) = e_3 \wedge e_4 \wedge e_2,\ \Delta(e_3) = e_1 \wedge e_4 \wedge e_2,\ \Delta(e_4) = e_1 \wedge e_3 \wedge e_2;$

(15) $\Delta(e_1) = e_3 \wedge e_4 \wedge e_2,\ \Delta(e_3) = e_1 \wedge e_4 \wedge e_2,\ \Delta(e_4) = e_3 \wedge e_1 \wedge e_2;$

(16) $\Delta(e_1) = e_4 \wedge e_3 \wedge e_2,\ \Delta(e_3) = e_4 \wedge e_1 \wedge e_2,\ \Delta(e_4) = e_1 \wedge e_3 \wedge e_2;$

(17) $\Delta(e_1) = e_4 \wedge e_3 \wedge e_2,\ \Delta(e_3) = e_4 \wedge e_1 \wedge e_2,\ \Delta(e_4) = e_3 \wedge e_1 \wedge e_2;$
(18) $\Delta(e_1) = e_4 \wedge e_3 \wedge e_2,\ \Delta(e_3) = e_1 \wedge e_4 \wedge e_2,\ \Delta(e_4) = e_1 \wedge e_3 \wedge e_2;$
(19) $\Delta(e_2) = e_4 \wedge e_3 \wedge e_1,\ \Delta(e_3) = e_4 \wedge e_2 \wedge e_1,\ \Delta(e_4) = e_3 \wedge e_2 \wedge e_1;$
(20) $\Delta(e_2) = e_4 \wedge e_3 \wedge e_1,\ \Delta(e_3) = e_2 \wedge e_4 \wedge e_1,\ \Delta(e_4) = e_3 \wedge e_2 \wedge e_1;$
(21) $\Delta(e_2) = e_4 \wedge e_3 \wedge e_1,\ \Delta(e_3) = e_4 \wedge e_2 \wedge e_1,\ \Delta(e_4) = e_2 \wedge e_3 \wedge e_1;$
(22) $\Delta(e_2) = e_3 \wedge e_4 \wedge e_1,\ \Delta(e_3) = e_2 \wedge e_4 \wedge e_1,\ \Delta(e_4) = e_3 \wedge e_2 \wedge e_1;$
(23) $\Delta(e_2) = e_3 \wedge e_4 \wedge e_1,\ \Delta(e_3) = e_2 \wedge e_4 \wedge e_1,\ \Delta(e_4) = e_2 \wedge e_3 \wedge e_1;$
(24) $\Delta(e_2) = e_3 \wedge e_4 \wedge e_1,\ \Delta(e_3) = e_4 \wedge e_2 \wedge e_1,\ \Delta(e_4) = e_3 \wedge e_2 \wedge e_1.$
and the six isomorphic 3-Lie coalgebras of the type C_e :
(1) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_1 \wedge e_3 \wedge e_4, \Delta(e_3) = e_1 \wedge e_2 \wedge e_4, \Delta(e_4) = e_1 \wedge e_2 \wedge e_3;$
(2) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_1 \wedge e_3 \wedge e_4,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4,\ \Delta(e_4) = e_2 \wedge e_1 \wedge e_3;$
(3) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_3 \wedge e_1 \wedge e_4, \Delta(e_3) = e_2 \wedge e_1 \wedge e_4, \Delta(e_4) = e_2 \wedge e_3 \wedge e_1;$
(4) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_1 \wedge e_4 \wedge e_3,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4,\ \Delta(e_4) = e_2 \wedge e_1 \wedge e_3;$
(5) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_4 \wedge e_1 \wedge e_3,\ \Delta(e_3) = e_2 \wedge e_4 \wedge e_1,\ \Delta(e_4) = e_2 \wedge e_1 \wedge e_3;$
(6) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_4 \wedge e_3 \wedge e_1,\ \Delta(e_3) = e_2 \wedge e_4 \wedge e_1,\ \Delta(e_4) = e_2 \wedge e_3 \wedge e_1,$
are incompatible with the 3-Lie algebra L_d. It follows the result. The proof is complete.

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