Mathematica Aeterna, Vol. 6, 2016, no. 4, 587 - 591

Bialgebra structures on simple 3-Lie algebra

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Abstract

The bialgebra structure on the finite dimensional simple 3-Lie algebra L_e over the field of complex numbers is studied. It is proved that there exist only three non-equivalent bialgebra structures on L_e, which are $(L_e, 0)$, (L_e, C_{c_3}, Δ_1) and (L_e, C_{c_3}, Δ_2).

2010 Mathematics Subject Classification: 17B05 17D30
Keywords: 3-Lie algebra, 3-Lie coalgebra, 3-Lie bialgebra.

1 Preliminaries

Authors in paper [1] provided 3-Lie algebrs, and then bialgebra structures on the 4-dimensional 3-Lie algebras L_b, L_c, L_d are discussed [2, 3, 4, 5, 6]. In this paper we discuss bialgebra structures on the finite dimensional simple 3-Lie algebra L_e [7] over the field of complex numbers.

W. Ling in paper [8] proved that there exists only one finite dimensional simple 3-Lie algebra over the complex field, that is the simple 4-dimensional 3-Lie algebra. Suppose L is a 4-dimensional vector space with a basis e_1, e_2, e_3, e_4. Then L is the simple 3-Lie algebra in the multiplication $\mu_e : L \wedge L \wedge L \to L$:

$$\mu_e(e_2, e_3, e_4) = e_1, \mu_e(e_1, e_3, e_4) = e_2, \mu_e(e_1, e_2, e_4) = e_3, \mu_e(e_1, e_2, e_3) = e_4,$$

and which is denoted by L_e.

A 3-Lie coalgebra (L, Δ) [1] is a vector space L with a linear mapping $\Delta : L \to L \otimes L \otimes L$ satisfying

(1) $Im(\Delta) \subset L \wedge L \wedge L$, (2)$(1-\omega_1-\omega_2-\omega_3)(1\otimes 1\otimes \Delta)\Delta = 0$,
where 1 is the identity, linear maps $\omega_1, \omega_2, \omega_3 : L^{\otimes 5} \to L^{\otimes 5}$ satisfying identities

$$\omega_1(x_1\otimes x_2\otimes x_3\otimes x_4\otimes x_5) = x_3\otimes x_4\otimes x_1\otimes x_2\otimes x_5,$$
$$\omega_2(x_1\otimes x_2\otimes x_3\otimes x_4\otimes x_5) = x_4\otimes x_5\otimes x_1\otimes x_2\otimes x_3,$$
$$\omega_3(x_1\otimes x_2\otimes x_3\otimes x_4\otimes x_5) = x_5\otimes x_3\otimes x_1\otimes x_2\otimes x_4.$$

A 3-Lie bialgebra [1] is a triple (L, μ, Δ) such that
(1) (L, μ) is a 3-Lie algebra with the multiplication $\mu : L \wedge L \wedge L \to L$,
(2) (L, Δ) is a 3-Lie coalgebra with $\Delta : L \to L \wedge L \wedge L$,
(3) Δ and μ satisfy the following identity, for $x, y, u, v, w \in L$,

$$\Delta\mu(x, y, z) = ad^{(3)}_\mu(x, y)\Delta(z) + ad^{(3)}_\mu(y, z)\Delta(x) + ad^{(3)}_\mu(z, x)\Delta(y),$$

where $ad^{(3)}_\mu(x, y)$, $ad^{(3)}_\mu(z, x)$, $ad^{(3)}_\mu(y, z) : L\otimes L\otimes L \to L\otimes L\otimes L$ are linear maps defined by (similar for $ad^{(3)}_\mu(z, x)$ and $ad^{(3)}_\mu(y, z)$)

$$ad^{(3)}_\mu(x, y)(u\otimes v\otimes w) = (ad_\mu(x, y)\otimes 1\otimes 1)(u\otimes v\otimes w)$$
$$+(1\otimes ad_\mu(x, y)\otimes 1)(u\otimes v\otimes w) + (1\otimes 1\otimes ad_\mu(x, y))(u\otimes v\otimes w)$$
$$= \mu(x, y, u)\otimes v\otimes w + u\otimes \mu(x, y, v)\otimes w + u\otimes v\otimes \mu(x, y, w).$$

Let V be a vector space, $\Delta : V \to V\otimes V\otimes V$ be a linear mapping, V^* be the dual space of V and Δ^* be the dual mapping of Δ. Then for all $f \in V^*, x_1, x_2, x_3 \in V$, $\langle \Delta(f), x_1\otimes x_2\otimes x_3\rangle = \langle f, \Delta^*(x_1, x_2, x_3)\rangle$.

We give the classification of 4-dimensional 3-Lie coalgebras.

Lemma 2.1 [1, 9] Let (L, Δ) be a 4-dimensional 3-Lie coalgebra with a basis e_1, e_2, e_3, e_4. Then L is isomorphic to one and only one of the following possibilities: C_a. (L, Δ_a) is trivial;

C_{b_1}. $\Delta_{b_1}(e^1) = e^2\wedge e^3\wedge e^4$; C_{b_2}. $\Delta_{b_2}(e^1) = e^1\wedge e^2\wedge e^3$;
C_{c_1}. $\Delta_{c_1}(e^1) = e^2\wedge e^3\wedge e^4$, $\Delta_{c_1}(e^2) = e^1\wedge e^3\wedge e^4$;
C_{c_2}. $\Delta_{c_2}(e^1) = \alpha e^2\wedge e^3\wedge e^4, \Delta_{c_2}(e^2) = e^2\wedge e^3\wedge e^4 + e^1\wedge e^3\wedge e^4$;
C_{c_3}. $\Delta_{c_3}(e^1) = e^1\wedge e^3\wedge e^4, \Delta_{c_3}(e^2) = e^2\wedge e^3\wedge e^4$;
C_d. $\Delta_d(e^1) = e^2\wedge e^3\wedge e^4, \Delta_d(e^2) = e^1\wedge e^3\wedge e^4, \Delta_d(e^3) = e^1\wedge e^2\wedge e^4$;
C_e. $\Delta_e(e^1) = e^2\wedge e^3\wedge e^4, \Delta_e(e^2) = e^1\wedge e^3\wedge e^4, \Delta_e(e^3) = e^1\wedge e^2\wedge e^4$,
$\Delta_e(e^4) = e^1\wedge e^2\wedge e^3$, where $\alpha \in F, \alpha \neq 0$.

For convenience, in the following, for a 3-Lie bialgebra (L, μ, Δ), if the 3-Lie algebra (L, μ) is the case (L, μ_e), and the 3-Lie coalgebra (L, Δ) is the case (L, Δ_{c_1}) for example, then the 3-Lie bialgebra (L, μ_e, Δ_{c_1}) is simply denoted by (L_e, C_{c_1}), which is called *the 3-Lie bialgebra of type* (L_e, C_{c_1}).

2 Bialgebra structures on L_e

For a given 3-Lie algebra L, in order to find all the 3-Lie bialgebra structures on L, we should find all the 3-Lie coalgebra structures on L which are compatible with the 3-Lie algebra L. Although a permutation of a basis of L

gives isomorphic 3-Lie coalgebra, but it may lead to the non-equivalent 3-Lie bialgebras.

Theorem The only bialgebra structures on the finite dimensional simple 3-Lie algebra L_e are (L_e, Δ_0), (where $\Delta_0 = 0 : L \to L^{\otimes 3}$), and (L_e, C_{c_3}). And the non-equivalent 3-Lie bialgebras of the type (L_e, C_{c_3}) are as follows:

$(L_e, C_{c_3}, \Delta_1)\Delta_1 e_1 = e_1 \wedge e_3 \wedge e_4, \Delta_1 e_2 = e_2 \wedge e_3 \wedge e_4$;

(L_e, C_{c_3}, Δ_2) $\Delta_2 e_1 = e_1 \wedge e_2 \wedge e_4, \Delta_2 e_3 = e_3 \wedge e_2 \wedge e_4$.

Proof It is clear that 3-Lie algebra L_e is compatible with coalgebra (L, Δ_0), where $\Delta_0 = 0 : L \to L^{\otimes 3}$. From Lemma 2.1, by means of permutating a basis of L, we obtain the twelve isomorphic 3-Lie coalgebras of type C_{c_3} as follows:

(1) $\Delta(e_1) = e_1 \wedge e_3 \wedge e_4, \Delta(e_2) = e_2 \wedge e_3 \wedge e_4$;
(2) $\Delta(e_1) = e_1 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_2 \wedge e_4 \wedge e_3$;
(3) $\Delta(e_1) = e_1 \wedge e_2 \wedge e_4, \Delta(e_3) = e_3 \wedge e_2 \wedge e_4$;
(4) $\Delta(e_1) = e_1 \wedge e_4 \wedge e_2, \Delta(e_3) = e_3 \wedge e_4 \wedge e_2$;
(5) $\Delta(e_1) = e_1 \wedge e_3 \wedge e_2,\ \Delta(e_4) = e_4 \wedge e_3 \wedge e_2$;
(6) $\Delta(e_1) = e_1 \wedge e_2 \wedge e_3, \Delta(e_4) = e_4 \wedge e_2 \wedge e_3$;
(7) $\Delta(e_2) = e_2 \wedge e_4 \wedge e_1, \Delta(e_3) = e_3 \wedge e_4 \wedge e_1$;
(8) $\Delta(e_2) = e_2 \wedge e_1 \wedge e_4,\ \Delta(e_3) = e_3 \wedge e_1 \wedge e_4$;
(9) $\Delta(e_2) = e_2 \wedge e_3 \wedge e_1, \Delta(e_4) = e_4 \wedge e_3 \wedge e_1$;
(10)$\Delta(e_2) = e_2 \wedge e_1 \wedge e_3, \Delta(e_4) = e_4 \wedge e_1 \wedge e_3$;
(11)$\Delta(e_3) = e_3 \wedge e_1 \wedge e_2,\ \Delta(e_4) = e_4 \wedge e_1 \wedge e_2$;
(12)$\Delta(e_3) = e_3 \wedge e_2 \wedge e_1, \Delta(e_4) = e_4 \wedge e_2 \wedge e_1$.

By a direct computation, the above twelve 3-Lie coalgebras are compatible with the 3-Lie algebra L_e, So we get 3-Lie bialgebras (L_e, C_{c_3}, Δ). By the following isomorphisms of the 4-Lie bialgebras

$(1) \to (5):\ f(e_1) = -e_4, f(e_2) = e_1, f(e_3) = e_2, f(e_4) = -e_3$;
$(1) \to (11):\ f(e_1) = e_3, f(e_2) = e_4, f(e_3) = e_1, f(e_4) = e_2$;
$(3) \to (9):\ f(e_1) = e_2, f(e_2) = e_1, f(e_3) = e_4, f(e_4) = -e_3$;
$(1) \to (2), (11) \to (12) : f(e_1) = e_1, f(e_2) = -e_2, f(e_3) = -e_3, f(e_4) = e_4$;
$(1) \to (8),\ (2) \to (7):\ f(e_1) = -e_3, f(e_2) = e_2, f(e_3) = e_1, f(e_4) = e_4$;
$(3) \to (4),\ (5) \to (6),\ (9) \to (10):$
$f(e_1) = -e_1, f(e_2) = -e_2, f(e_3) = e_3, f(e_4) = e_4$;

we get the non-equivalent 3-Lie bialgebras of type (L_e, C_{c_3}) are (L_e, C_{c_3}, Δ_1) and (L_e, C_{c_3}, Δ_2).

Now we prove that there does not exist 3-Lie bialgebra of type (L_e, C_b).

First, we prove that there does not exist 3-Lie bialgebra of type (L_e, C_{b_1}). By Lemma 2.1, by means of permutating a basis of L, we obtain the eight isomorphic 3-Lie coalgebras of type C_{b_1}:

(1) $\Delta\ (e_1) = e_2 \wedge e_3 \wedge e_4$; (2) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3, ;$ (3) $\Delta(e_2) = e_1 \wedge e_4 \wedge e_3$;
(4) $\Delta(e_2) = e_1 \wedge e_3 \wedge e_4$; (5) $\Delta(e_3) = e_1 \wedge e_2 \wedge e_4$; (6) $\Delta(e_3) = e_1 \wedge e_4 \wedge e_2$;

(7) $\Delta(e_4) = e_1 \wedge e_2 \wedge e_3$; (8) $\Delta(e_4) = e_2 \wedge e_1 \wedge e_3$.
By a direct computation, the above 3-Lie coalgebras are incompatible with the 3-Lie algebra L_e. Therefore there does not exist 3-Lie bialgebra (L_d, C_{b_1}).

Second, from Lemma 2.1, we have twenty four isomorphic 3-Lie coalgebras of the type C_{b_2}:

(1)$\Delta(e_1) = e_1 \wedge e_2 \wedge e_3$; (2)$\Delta(e_1) = e_1 \wedge e_2 \wedge e_4$; (3)$\Delta(e_1) = e_1 \wedge e_3 \wedge e_4$;

(4)$\Delta(e_1) = e_1 \wedge e_3 \wedge e_2$; (5)$\Delta(e_1) = e_1 \wedge e_4 \wedge e_2$; (6)$\Delta(e_1) = e_1 \wedge e_4 \wedge e_3$;

(7)$\Delta(e_2) = e_2 \wedge e_1 \wedge e_3$; (8)$\Delta(e_2) = e_2 \wedge e_1 \wedge e_4$; (9)$\Delta(e_2) = e_2 \wedge e_3 \wedge e_4$;

(10)$\Delta(e_2) = e_2 \wedge e_3 \wedge e_1$; (11)$\Delta(e_2) = e_2 \wedge e_4 \wedge e_1$; (12)$\Delta(e_2) = e_2 \wedge e_4 \wedge e_3$;

(13)$\Delta(e_3) = e_3 \wedge e_1 \wedge e_2$; (14)$\Delta(e_3) = e_3 \wedge e_1 \wedge e_4$; (15)$\Delta(e_3) = e_3 \wedge e_2 \wedge e_4$;

(16)$\Delta(e_3) = e_3 \wedge e_2 \wedge e_1$; (17)$\Delta(e_3) = e_3 \wedge e_4 \wedge e_1$; (18)$\Delta(e_3) = e_3 \wedge e_4 \wedge e_2$;

(19)$\Delta(e_4) = e_4 \wedge e_1 \wedge e_2$; (20)$\Delta(e_4) = e_4 \wedge e_1 \wedge e_3$; (21)$\Delta(e_4) = e_4 \wedge e_2 \wedge e_3$;
(22)$\Delta(e_4) = e_4 \wedge e_2 \wedge e_1$; (23)$\Delta(e_4) = e_4 \wedge e_3 \wedge e_1$; (24)$\Delta(e_4) = e_4 \wedge e_3 \wedge e_2$.

By a direct computation, the 3-Lie coalgebras of the type C_{b_2} are incompatible with the 3-Lie algebra L_e. Therefore, there does not exist 3-Lie bialgebras of type (L_e, C_b).

Thirdly, we prove that there does not exist 3-Lie bialgebra of type (L_e, C_e). By the similar discussion, we have six isomorphic 3-Lie coalgebras of the type C_e :

(1) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = e_1 \wedge e_3 \wedge e_4, \Delta(e_3) = e_1 \wedge e_2 \wedge e_4,$
$\Delta(e_4) = e_1 \wedge e_2 \wedge e_3$;
(2) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_1 \wedge e_3 \wedge e_4,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4,$
$\Delta(e_4) = e_2 \wedge e_1 \wedge e_3$;
(3)$\Delta(e_1) = e_2 \wedge e_3 \wedge e_4,\ \Delta(e_2) = e_3 \wedge e_1 \wedge e_4,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4,$
$\Delta(e_4) = e_2 \wedge e_3 \wedge e_1$;
(4) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_1 \wedge e_4 \wedge e_3,\ \Delta(e_3) = e_2 \wedge e_1 \wedge e_4,$
$\Delta(e_4) = e_2 \wedge e_1 \wedge e_3$;
(5) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_4 \wedge e_1 \wedge e_3,\ \Delta(e_3) = e_2 \wedge e_4 \wedge e_1,$
$\Delta(e_4) = e_2 \wedge e_1 \wedge e_3$;
(6) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3,\ \Delta(e_2) = e_4 \wedge e_3 \wedge e_1,\ \Delta(e_3) = e_2 \wedge e_4 \wedge e_1,$
$\Delta(e_4) = e_2 \wedge e_3 \wedge e_1,$

are incompatible with the 3-Lie algebra L_e.

Lastly, by completely similar discussion, we can prove that there do not exist 3-Lie bialgebras of types (L_e, C_{c_1}), (L_e, C_{c_2}) and (L_e, C_d). We omit the similar computation. The proof is complete.

Acknowledgements

The first author (R.-P. Bai) was supported in part by the Natural Science Foundation (11371245) and the Natural Science Foundation of Hebei Province (A2014 201006).

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Received: August, 2016