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The construction of Hom-Novikov superalgebras

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Abstract

We study a twisted generalization of Novikov superalgebras, called Hom-Novikov superalgebras. It is shown that two classes of Hom-Novikov superalgebras can be constructed from Hom-supercommutative algebras together with derivations and Hom-Novikov superalgebras with Rota-Baxter operators, respectively.

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1 Introduction

Novikov algebras were firstly introduced in the study of Hamiltonian operators concerning integrability of certain nonlinear partial differential equations.[2] Yau in [3] introduced Hom-Novikov algebras, in which the two defining identities are twisted by a linear map. It turned out that Hom-Novikov algebras can be constructed from Novikov algebras, commutative Hom-associative algebras and Hom-Lie algebras along with some suitable linear maps. Later, Zhang, Hou and Bai in [4] defined a Hom-Novikov superalgebra as a twisted generalization of Novikov superalgebras.

The purpose of this paper is to consider the realization of Hom-Novikov superalgebras. It is shown that two classes of Hom-Novikov superalgebras can be constructed from Hom supercommutative algebras together with derivations and Hom-Novikov superalgebras with Rota-Baxter operators, respectively.

Throughout this paper $\mathbf{F}$ denotes an arbitrary field.

2 Main Results

Let $(A, \cdot)$ be an algebra over field $\mathbf{F}$. A is said to be a superalgebra if the underlying vector space of A is $\mathbb{Z}_2$-graded (i.e., $A = A_{\bar{0}} \oplus A_{\bar{1}}$, where $A_{\bar{0}}$ and $A_{\bar{1}}$

are vector subspaces of A) and $A_\alpha \cdot A_\beta \subset A_{\alpha+\beta}$, $\forall \alpha, \beta \in \mathbb{Z}_2$. An element $x \in A$ is called homogeneous if $x \in A_{\bar{0}} \cup A_{\bar{1}}$. In this work, all elements are supposed to be homogeneous unless otherwise stated. For a homogeneous element x we shall use the standard notation $|x| \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ to indicate its degree.

Definition 2.1 [1] *A Hom-associative superalgebra consists of a $\mathbb{Z}_2$-graded vector space A, a linear self-map α and an even bilinear map $\mu : A \times A \to A$, satisfying*

$$\alpha(xy) = \alpha(x)\alpha(y) \quad (multiplicativity)$$

and

$$(xy)\alpha(z) = \alpha(x)(yz) \quad (Hom\!-\!associativity),$$

for $x, y, z \in A$.

Definition 2.2 [4] *A Hom-Novikov superalgebra is a triple (A, μ, α) consisting of a $\mathbb{Z}_2$-graded vector space A, an even bilinear map $\mu : A \times A \to A$ and an even linear map $\alpha : A \to A$ satisfying*

$$\alpha(xy) = \alpha(x)\alpha(y)\,(multiplicativity), \tag{1}$$

$$(xy)\alpha(z) - \alpha(x)(yz) = (-1)^{|x||y|}((yx)\alpha(z) - \alpha(y)(xz)), \tag{2}$$

$$(xy)\alpha(z) = (-1)^{|y||z|}(xz)\alpha(y). \tag{3}$$

Definition 2.3 *Let (A, μ, α) be a Hom-Novikov superalgebra, which is called involutive if α is an involution, i.e., $\alpha^2 = \mathrm{id}$.*

Proposition 2.4 *If (A, μ, α) is an involutive Hom-Novikov superalgebra, then $(A, \alpha \circ \mu)$ is a Novikov superalgebra.*

Proof. For convenience, we write $x * y = \alpha(xy)$, for all $x, y \in A$. Hence, it needs to show

$$(x * y) * z - x * (y * z) = (-1)^{|x||y|}((y * x) * z - y * (x * z)), \tag{4}$$

$$(x * y) * z = (-1)^{|y||z|}(x * z) * y, \tag{5}$$

for all $x, y, z \in A$. Since (A, μ, α) is an involutive Hom-Novikov superalgebra, we have

$$\begin{aligned}(x * y) * z &= \alpha(\alpha(xy)z) = \alpha^2(xy)\alpha(z) \\ &= (xy)\alpha(z) = (-1)^{|y||z|}(xz)\alpha(y) = (-1)^{|y||z|}(x * z) * y.\end{aligned}$$

Furthermore,

$$\begin{aligned}&(x * y) * z - x * (y * z) = \alpha(\alpha(xy)z) - \alpha(x\alpha(yz)) = (xy)\alpha(z) - \alpha(x)(yz) \\ = \; &(-1)^{|x||y|}((yx)\alpha(z) - \alpha(y)(xz)) = (-1)^{|x||y|}((y * x) * z - y * (x * z)),\end{aligned}$$

which proves Equation (4) and the proposition. □

Proposition 2.5 *Let (A,μ,α) be a Hom-Novikov superalgebra. Then $(A,\alpha\circ\mu,\alpha^2)$ is a Hom-Novikov superalgebra.*

Proof. For convenience, we write $x*y=\alpha(xy)$, for all $x,y\in A$. Hence, we need to prove

$$(x*y)*\alpha^2(z)-\alpha^2(x)*(y*z)=(-1)^{|x||y|}((y*x)*\alpha^2(z)-\alpha^2(y)*(x*z)), \quad (6)$$

$$(x*y)*\alpha^2(z)=(-1)^{|y||z|}(x*z)*\alpha^2(y), \quad (7)$$

for all $x,y,z\in A$. Since (A,μ,α) is a Hom-Novikov superalgebra, we have

$$(x*y)*\alpha^2(z)=\alpha^2((xy)\alpha(z))=(-1)^{|y||z|}\alpha^2((xz)\alpha(y))=(-1)^{|y||z|}(x*z)*\alpha^2(y).$$

Furthermore,

$$\begin{aligned} & (x*y)*\alpha^2(z)-\alpha^2(x)*(y*z)=\alpha^2((xy)\alpha(z)-\alpha(x)(yz)) \\ = \; & (-1)^{|x||y|}\alpha^2((yx)\alpha(z)-\alpha(y)(xz))=(-1)^{|x||y|}((y*x)*\alpha^2(z)-\alpha^2(y)*(x*z)), \end{aligned}$$

which proves Equation (6) and the proposition. □

Theorem 2.6 *Let (A,μ,α) be a Hom-supercommutative algebra and $D: A\to A$ be an even derivation such that $D\alpha=\alpha D$. Then $(A,*,\alpha)$ is a Hom-Novikov superalgebra, where $*$ is defined by*

$$x*y=\mu(x,D(y))=xD(y), \quad (8)$$

Proof. The multiplicativity of α with respect to $*$ in (8) follows from the multiplicativity of α with respect to μ and the hypothesis $D\alpha=\alpha D$. Next we check (2)

$$\begin{aligned} & (x*y)*\alpha(z)-\alpha(x)*(y*z)=(xD(y))D(\alpha(z))-\alpha(x)D(yD(z)) \\ = \; & (xD(y))\alpha(D(z))-\alpha(x)(D(y)D(z))-\alpha(x)(yD^2(z))=-(xy)\alpha(D^2(z)). \end{aligned}$$

The last two equalities follow from Hom-associativity. On the other hand,

$$\begin{aligned} & (-1)^{|x||y|}((y*x)*\alpha(z)-\alpha(y)*(x*z)) \\ = \; & (-1)^{|x||y|}((yD(x))D(\alpha(z))-\alpha(y)D(xD(z))) \\ = \; & (-1)^{|x||y|}((yD(x))\alpha(D(z))-\alpha(y)(D(x)D(z))-\alpha(y)(xD^2(z))) \\ = \; & (-1)^{|x||y|}(\alpha(y)(D(x)D(z))-\alpha(y)(D(x)D(z))-\alpha(y)(xD^2(z))) \\ = \; & -(-1)^{|x||y|}\alpha(y)(xD^2(z))=-(-1)^{|x||y|}(yx)\alpha(D^2(z))=-(xy)\alpha(D^2(z)). \end{aligned}$$

Futhermore, we have

$$\begin{aligned} & (x*y)*\alpha(z)=(xD(y))\alpha(D(z))=\alpha(x)(D(y)D(z)) \\ = \; & (-1)^{|y||z|}\alpha(x)(D(z)D(y))=(-1)^{|y||z|}(xD(z))\alpha(D(y))=(-1)^{|y||z|}(x*z)*\alpha(y). \end{aligned}$$

Consequently, we prove the theorem. □

Definition 2.7 *Let $(A, *, \alpha)$ be a Hom-superalgebra and let $\lambda \in \mathbf{F}$. If a linear map $P : A \to A$ satisfies*

$$P(x) * P(y) = P(P(x) * y + x * P(y) + \lambda x * y), \ \ \forall x, y \in A,$$

then P is called a Rota-Baxter operator of weight λ and $(A, *, \alpha, P)$ is called a Rota-Baxter Hom-superalgebra of weight λ.

Theorem 2.8 *Let $(A, *, \alpha, P)$ be a Rota-Baxter Hom-Novikov superalgebra of weight λ and P an even linear map. Assume that α and P commute. Then $(A, \circ, \alpha, P)$ is a Hom-Novikov superalgebra, where the multiplication $\circ$ is defined as*

$$x \circ y := P(x) * y + x * P(y) + \lambda x * y, \ \ \forall x, y \in A.$$

Proof. The multiplicativity of α with respect to $\circ$ follows from the multiplicativity of α with respect to $*$ and the hypothesis $P\alpha = \alpha P$. For any $x, y, z \in A$, we have,

$$\begin{aligned}
& (x \circ y) \circ \alpha(z) - \alpha(x) \circ (y \circ z) \\
= \ & (P(x) * P(y)) * \alpha(z) + (P(x) * y) * \alpha(P(z)) + (x * P(y)) * \alpha(P(z)) \\
& + \lambda(x * y) * \alpha(P(z)) + \lambda(P(x) * y) * \alpha(z) + \lambda(x * P(y)) * \alpha(z) \\
& + \lambda^2(x * y) * \alpha(z) - \alpha(P(x)) * (P(y) * z) - \alpha(P(x)) * (y * P(z)) \\
& - \lambda\alpha(P(x)) * (y * z) - \alpha(x) * (P(y) * P(z)) - \lambda\alpha(x) * (P(y) * z) \\
& - \lambda\alpha(x) * (y * P(z)) - \lambda^2\alpha(x) * (y * z) \\
= \ & (P(x) * P(y)) * \alpha(z) - \alpha(P(x)) * (P(y) * z) + (P(x) * y) * \alpha(P(z)) \\
& - \alpha(P(x)) * (y * P(z)) + (x * P(y)) * \alpha(P(z)) - \alpha(x) * (P(y) * P(z)) \\
& + \lambda(x * y) * \alpha(P(z)) - \lambda\alpha(x) * (y * P(z)) + \lambda(P(x) * y) * \alpha(z) \\
& - \lambda\alpha(P(x)) * (y * z) + \lambda(x * P(y)) * \alpha(z) - \lambda\alpha(x) * (P(y) * z) \\
& + \lambda^2(x * y) * \alpha(z) - \lambda^2\alpha(x) * (y * z) \\
= \ & (-1)^{|x||y|}\Big((P(y) * P(x)) * \alpha(z) - \alpha(P(y)) * (P(x) * z) + (y * P(x)) * \alpha(P(z)) \\
& - \alpha(y) * (P(x) * P(z)) + (P(y) * x) * \alpha(P(z)) - \alpha(P(y)) * (x * P(z)) \\
& + \lambda(y * x) * \alpha(P(z)) - \lambda\alpha(y) * (x * P(z)) + \lambda(y * P(x)) * \alpha(z) - \lambda\alpha(y) * (P(x) * z) \\
& + \lambda(P(y) * x) * \alpha(z) - \lambda\alpha(P(y)) * (x * z) + \lambda^2(y * x) * \alpha(z) - \lambda^2\alpha(y) * (x * z)\Big).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& (-1)^{|x||y|}\Big((y \circ x) \circ \alpha(z) - \alpha(y) \circ (x \circ z)\Big) \\
= \ & (-1)^{|x||y|}\Big((P(y) * P(x)) * \alpha(z) - \alpha(P(y)) * (P(x) * z) + (P(y) * x) * \alpha(P(z))
\end{aligned}$$

$$-\alpha(P(y))*(x*P(z))+(y*P(x))*\alpha(P(z))-\alpha(y)*(P(x)*P(z))$$
$$+\lambda(y*x)*\alpha(P(z))-\lambda\alpha(y)*(x*P(z))+\lambda(P(y)*x)*\alpha(z)-\lambda\alpha(P(y))*(x*z)$$
$$+\lambda(y*P(x))*\alpha(z)-\lambda\alpha(y)*(P(x)*z)+\lambda^2(y*x)*\alpha(z)-\lambda^2\alpha(y)*(x*z)\Big).$$

Furthermore, on the one hand, we have

$$\begin{aligned}
&(x\circ y)\circ\alpha(z)\\
=\;& P(P(x)*y+x*P(y)+\lambda x*y)*\alpha(z)+(P(x)*y+x*P(y))*P(\alpha(z))\\
&+(\lambda x*y)*P(\alpha(z))+\lambda(P(x)*y+x*P(y)+\lambda x*y)*\alpha(z)\\
=\;& (P(x)*P(y))*\alpha(z)+(P(x)*y)*\alpha(P(z))+(x*P(y))*\alpha(P(z))\\
&+\lambda(x*y)*\alpha(P(z))+\lambda(P(x)*y)*\alpha(z)+\lambda(x*P(y))*\alpha(z)+\lambda^2(x*y)*\alpha(z)\\
=\;& (-1)^{|y||z|}\Big((P(x)*z)*\alpha(P(y))+(P(x)*P(z))*\alpha(y)+(x*P(z))*\alpha(P(y))\\
&+\lambda(x*P(z))*\alpha(y)+\lambda(P(x)*z)*\alpha(y)+\lambda(x*z)*\alpha(P(y))+\lambda^2(x*z)*\alpha(y)\Big).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&(-1)^{|y||z|}\Big((x\circ z)\circ\alpha(y)\Big)\\
=\;& (-1)^{|y||z|}\Big((P(x)*P(z))*\alpha(y)+(P(x)*z)*\alpha(P(y))+(x*P(z))*\alpha(P(y))\\
&+\lambda(x*z)*\alpha(P(y))+\lambda(P(x)*z)*\alpha(y)+\lambda(x*P(z))*\alpha(y)+\lambda^2(x*z)*\alpha(y)\Big).
\end{aligned}$$

Hence, the conclusion holds. □

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References

[1] Ammar, F.; Makhlouf, A., Hom-Lie superalgebras and Hom-Lie admissible superalgebras. J. Algebra, 324 (2010), 1513–1528..

[2] Gel'fand, I.; Dorfman, I., Hamiltonian operators and algebraic structures related to them. Funktsional. Anal. i Priložen., 13 (1979), 13–30.

[3] Yau, D., Hom-Novikov algebras. J. Phys. A: Math. Theor., 44 (2011), 085202, 20pp.

[4] Zhang, R.; Hou, D.; Bai, C., A Hom-version of the affinizations of Balinskii-Novikov and Novikov superalgebras. J. Math. Phys., 52 (2011), 023505, 19 pp.

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