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Properties of Expected Value for Uncertain Variables

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Abstract

In order to demonstrate some theories of mathematical expectation applied under uncertain environment, we need to study properties of expected value. Based on uncertainty theory, some properties and theorems of expected value are investigated. Furthermore, the concept of sublinear expectation is obtained, which is a special case of uncertain expectation.

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1 Introduction

When non-determinate phenomena in nature is neither randomness nor fuzziness, we can not solve this phenomena by subjective probability or fuzzy set theory. In order to describe these uncertainty phenomena, Liu [3] founded uncertainty theory in 2007 whose most fundamental concept is uncertain measure that satisfying normality, duality, subadditivity, and product axioms. Then some properties of uncertain measure was researched by Gao [2]. To represent quantities with uncertainty, Liu [3] introduced uncertain variable which is a measurable function from an uncertain space to the set of real numbers. Thereafter, some convergence theorems of uncertain sequence were investigated by You [16] in 2009. Liu [5] presented uncertainty programming with applications to machine scheduling, vehicle routing problem, project scheduling problem and so on. The uncertain risk analysis which is proposed by Liu

[10] is a tool to quantify risk via uncertainty theory. In addition, uncertain calculus (Liu [6] and Yao [18]) is a branch of mathematics that deals with differentiation and integral of uncertain process (Liu [4] and Yao [17]). Liu [13] also investigated the property of uncertain process. To date, many studies of uncertainty theory were put forward. For instance, uncertain inference and uncertain set was proposed by Liu [12], uncertain entailment and modus ponens was given by Liu [8] and uncertain logic was presented by Li and Liu [7], etc. Besides these, uncertain differential equation was introduced by Yao and Chen [18] and as an application of which is uncertain finance (Chen [1]).

Expectation is the foundation of uncertain programming, therefore Liu [3] give the definition of expected value. Then, Liu and Ha [11] proved the expected value of monotone function of uncertain variable is just a Lebesgue-Stieltjes integral of the function with respect to its uncertainty distribution, and give some useful expressions of expected value of function of uncertain variables. Furthermore, it is applied to some portfolio models [15], such as expected utility-variance model, expected utility-VaR model, expected utility-TVaR model. In order to make expected value have a wider application, the theorems and properties of expected value will be expanded in this paper.

The structure of this paper is organized as follows. In section 2, some basic definitions and theorems of uncertainty theory are recalled. Section 3 investigate some theorems and properties about the expected value of uncertain variables. At the end of this paper, a brief summary is given.

2 Preliminary Notes

In this section, we presented some definitions and theorems in uncertain environment within the framework of uncertainty theory which will be used in this paper.

2.1 Uncertain theory

Let Γ be a nonempty set, and let $\mathcal{L}$ be a $\sigma-$algebra over Γ. A number $\mathcal{M}\{\Lambda\}$ indicates the level that each element $\Lambda \in \mathcal{L}$ (which is called an event) will occur. Liu [3] proposed the set function $\mathcal{M}$, which is called uncertain measures if it satisfies the following three axioms:

Axiom 1 *(Normality)* $\mathcal{M}\{\Gamma\} = 1$.

Axiom 2 *(Self-Duality)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$, *for any event* Λ.

Axiom 3 *(Subadditivity) For every countable sequence of events* $\{\Lambda_i\}$, *we have*

$$\mathcal{M}\{\bigcup_{i=1}^{\infty}\Lambda_i\} \le \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Next, the definition of uncertain space is introduced.

Definition 2.1 *(Liu [6]) Let* Γ *be a nonempty set, let* $\mathcal{L}$ *be a* $\sigma-$*algebra over* Γ*, and let* $\mathcal{M}$ *be an uncertain measure. Then the triplet* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called an uncertainty space.*

The properties of uncertain measure is recalled in the following theorem.

Theorem 2.1 *(Monotonicity, [6]) Uncertain measure* $\mathcal{M}$ *is a monotone increasing set function. That is, for any events* $\Lambda_1 \subset \Lambda_2$*, we have* $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$.

Definition 2.2 *(Liu [3]) An uncertain variable is a function* ξ *from an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers such that* $\{\xi \in B\}$ *is an event for any set* B *of real numbers.*

Definition 2.3 *(Liu [3]) Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be uncertain variables, and let* f *be a real-valued measurable function. Then* $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ *is an uncertain variable defined by*

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \cdots, \xi_n(\gamma)), \ \forall \gamma \in \Gamma.$$

Definition 2.4 *(Liu [6]) The uncertain variables* $\xi_1, \xi_2, \cdots, \xi_n$ *are said to be independent if*

$$\mathcal{M}\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\} = \bigwedge_{i=1}^{n} \mathcal{M}\{(\xi_i \in B_i)\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ *of real numbers.*

Definition 2.5 *(Liu [3]) The uncertainty distribution* Φ *of an uncertain variable* ξ *is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

After the definition of uncertain distribution, we recalled the concept of regular uncertain distribution.

Definition 2.6 *(Liu [9]) An uncertain distribution* $\Phi(x)$ *is said to be regular if it is a continuous and strictly increasing function with respect to* x *at which* $0 < \Phi(x) < 1$*, and*

$$\lim_{x \to -\infty} \Phi(x) = 0, \lim_{x \to +\infty} \Phi(x) = 1.$$

Theorem 2.2 *(Liu [9]) Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be independent uncertain variables with regular uncertain distributions* $\Phi_1, \Phi_2, \cdots, \Phi_n$*, respectively. If* f *is a strictly increasing function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)).$$

2.2 Expected value of uncertain variable

The definition and properties of expected value in uncertainty theory are introduced in the following section.

Definition 2.7 *(Liu [3]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}\mathrm{d}x,$$

provided that at least one of the two integrals is finite.

Theorem 2.3 *(Liu [9]) Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

Theorem 2.4 *(Liu [9]) Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

By Theorem 2.4, we can obtain two conclusions as follows.

Remark 2.1 Let ξ, η be independent uncertain variables with finite expected values. Then

$$E[\xi + \eta] = E[\xi] + E[\eta].$$

Remark 2.2 Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with finite expected values. Then we have

$$E[\sum_{i=1}^{n} \xi_i] = \sum_{i=1}^{n} E[\xi_i].$$

Definition 2.8 *(Liu [3]) Let ξ be an uncertain variable with finite expected value e. Then the variance of ξ is*

$$V[\xi] = E[(\xi - e)^2].$$

Remark 2.3 Definition 2.8 tells us that the variance is just the expected value of $(\xi - e)^2$. Since $(\xi - e)^2$ is a nonnegative uncertain variable, we also have

$$V[\xi] = \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq x\}\mathrm{d}x.$$

Theorem 2.5 *(Peng [14]) We call $\rho(.)$ is a coherent risk measure if it satisfying the following four properties:*

(1) Monotonicity: *If $X \geq Y$ then $\rho(X) \leq \rho(Y)$.*

(2) Constant preserving: $\rho(l) = -\rho(-1) = -1$.

(3) Sub-additivity: *For each $X, Y \in \mathcal{H}$, $\rho(X + Y) \leq \rho(X) + \rho(Y)$.*

(4) Positive homogeneity: *$\rho(\lambda X) = \lambda\rho(X)$ for $\lambda \geq 0$.*

3 Main Results

The purpose of this section is to investigate some new theorems of expected value and give the concept of sublinear expectation.

Let we introduce an inequality on expected value.

Theorem 3.1 *(Constant preserving) Let ξ be an uncertain variable, and let a be any real number. Then we have $E[a] = a$.*

Proof: Let $\xi = a$. Then

$$\Phi(\alpha) = \begin{cases} 1, & \xi \geq \alpha, \\ 0, & \xi < \alpha. \end{cases}$$

If $a \geq 0$, we have

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x \\ &= \int_0^a (1-0)\mathrm{d}x + \int_a^{+\infty} (1-1)\mathrm{d}x - \int_{\infty}^0 0\mathrm{d}x \\ &= \int_0^a 1\mathrm{d}x \\ &= a. \end{aligned}$$

If $a < 0$,

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x \\ &= \int_0^{+\infty} (1-1)\mathrm{d}x - \int_{-\infty}^a (0)\mathrm{d}x - \int_a^0 1\mathrm{d}x \\ &= -\int_a^0 1\mathrm{d}x \\ &= a. \end{aligned}$$

Therefore, $E[a] = a$. The theorem is proved.

Definition 3.1 *Let ξ and η be uncertain variables defined on uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. We say $\xi \leq \eta$, if $\xi(\gamma) \leq \eta(\gamma)$ for almost all $\gamma \in \Gamma$.*

Definition 3.1 will be immediately applied to obtain the following conclusion in Theorem 3.2.

Theorem 3.2 *(Monotonicity) Let ξ_1, ξ_2 be uncertain variables with finite expected values. If $\xi_1 \leq \xi_2$, then we have $E[\xi_1] \leq E[\xi_2]$.*

Proof: For any x, if $\xi_1 \le \xi_2$, then $\{\xi_2 \le x\} \subset \{\xi_1 \le x\}$. It follows from Theorem 2.1 that $\mathcal{M}\{\xi_2 \le x\} \le \mathcal{M}\{\xi_1 \le x\}$. According to Axiom 2, we get

$$\mathcal{M}\{\xi_2 > x\} = 1 - \mathcal{M}\{\xi_2 \le x\},\ \mathcal{M}\{\xi_1 > x\} = 1 - \mathcal{M}\{\xi_1 \le x\},$$

therefore,

$$\mathcal{M}\{\xi_2 > x\} \ge \mathcal{M}\{\xi_1 > x\}.$$

By using the definition of expected value, we obtain

$$\begin{aligned} E[\xi_1] - E[\xi_2] &= (\int_0^{+\infty} \mathcal{M}\{\xi_1 \ge x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi_1 < x\}\mathrm{d}x) \\ &\quad - (\int_0^{+\infty} \mathcal{M}\{\xi_2 \ge x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi_2 < x\}\mathrm{d}x) \\ &= (\int_0^{+\infty} \mathcal{M}\{\xi_1 \ge x\}\mathrm{d}x - \int_0^{+\infty} \mathcal{M}\{\xi_2 \ge x\}\mathrm{d}x) \\ &\quad - (\int_{-\infty}^0 \mathcal{M}\{\xi_2 < x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi_1 < x\}\mathrm{d}x) \\ &\le 0, \end{aligned}$$

which proves the inequality.

As we know, if ξ_1, ξ_2 are independent uncertain variables, then $E[\xi_1+\xi_2] = E[\xi_1]+E[\xi_2]$. What's the connection between the expected values for the sums of uncertain variables and the sums of expected values if the independence is not assumed? Now the result can be illustrated by the following example.

Example 3.1 Take $(\Gamma, \mathcal{L}, \mathcal{M})$ be $\{\gamma_1, \gamma_2, \gamma_3\}$ with $\mathcal{M}\{\gamma_1\} = 0.5$, $\mathcal{M}\{\gamma_2\} = 0.7$ and $\mathcal{M}\{\gamma_3\} = 0.1$. It follows from Axiom 3 that $\mathcal{M}\{\gamma_1, \gamma_2\} = 0.9$, $\mathcal{M}\{\gamma_1, \gamma_3\} = 0.3$, $\mathcal{M}\{\gamma_2, \gamma_3\} = 0.5$. Define two uncertain variables as follows,

$$\xi(\gamma) = \begin{cases} 0, & if\ \gamma = \gamma_1 \\ 1, & if\ \gamma = \gamma_2 \\ 3, & if\ \gamma = \gamma_3, \end{cases} \quad \eta(\gamma) = \begin{cases} 4, & if\ \gamma = \gamma_1 \\ 2, & if\ \gamma = \gamma_2 \\ 5, & if\ \gamma = \gamma_3. \end{cases}$$

Note that ξ and η are not independent, and the sum is

$$(\xi + \eta)(\gamma) = \begin{cases} 4, & if\ \gamma = \gamma_1 \\ 3, & if\ \gamma = \gamma_2 \\ 8, & if\ \gamma = \gamma_3. \end{cases}$$

According to the definition of expected value, we get

$$\begin{aligned} E[\xi] &= \int_0^1 \mathcal{M}\{\xi \ge x\}\mathrm{d}x + \int_1^3 \mathcal{M}\{\xi \ge x\}\mathrm{d}x - 0 \\ &= \int_0^1 0.5\mathrm{d}x + \int_1^3 0.1\mathrm{d}x = 0.7. \end{aligned}$$

It is easy to verify that $E[\eta] = 2.7$, and $E[\xi + \eta] = 3.7$. Thus we have

$$E[\xi] + E[\eta] < E[\xi + \eta].$$

If the two uncertain variables are defined by

$$\xi(\gamma) = \begin{cases} 1, \ if \ \gamma = \gamma_1 \\ 2, \ if \ \gamma = \gamma_2 \\ 3, \ if \ \gamma = \gamma_3, \end{cases} \quad \eta(\gamma) = \begin{cases} 0, \ if \ \gamma = \gamma_1 \\ 4, \ if \ \gamma = \gamma_2 \\ 2, \ if \ \gamma = \gamma_3. \end{cases}$$

Then

$$(\xi + \eta)(\gamma) = \begin{cases} 1, \ if \ \gamma = \gamma_1 \\ 6, \ if \ \gamma = \gamma_2 \\ 5, \ if \ \gamma = \gamma_3. \end{cases}$$

It is easy to verify that $E[\xi] = 1.6, E[\eta] = 2.4$, and $E[\xi + \eta] = 3.7$. Thus we have

$$E[\xi] + E[\eta] > E[\xi + \eta].$$

That is to say, $E[\xi] + E[\eta] < E[\xi + \eta]$, $E[\xi] + E[\eta] > E[\xi + \eta]$.

By Theorem 2.4, Theorem 3.1, Theorem 3.2 and Example 3.1, we obtain the definition of sublinear expectation.

Definition 3.2 *Let ξ and η be uncertain variables and a be any number. We call* E *a sublinear expectation if it satisfies*

(1) Monotonicity: $\mathrm{E}[\xi] \geq \mathrm{E}[\eta]$, *if* $\xi \geq \eta$.

(2) Constant preserving: $\mathrm{E}[a] = a$ *for* $c \in R$.

(3) Sub-additivity: $\mathrm{E}[\xi + \eta] \leq \mathrm{E}[\xi] + \int E[\eta]$.

(4) Positive homogeneity: $\mathrm{E}[\lambda\xi] = \lambda\mathrm{E}[\xi]$ *for* $\lambda \geq 0$.

Let Ω be the set of uncertain variables and let H be a linear space of real valued functions defined on Ω. Then the triple (Ω, H, E) is called a sublinear expectation space.

Remark 3.1 Sublinear expectation is a special case of uncertain expectation, that is expectation meet the condition of Sub-additivity.

Remark 3.2 Let ξ be an uncertain variable. By Theorem 2.5, we known that if E is a Sublinear expectation, then $\rho(\xi) = \mathrm{E}[-\xi/r]$ is a coherent risk measure. Conversely, if ρ is a coherent risk measure, then $\mathrm{E}[\xi] = \rho(-\xi \cdot r)$ is a Sublinear expectation, where r is the total rate of return on a reference instrument. Thus Sublinear expectation can be also used for measuring risk.

Next, we will discuss the other properties of expectation.

Theorem 3.3 *Let f, g be nonnegative comonotonic functions. Then for any uncertain variable ξ, we have*

$$E[f(\xi) + g(\xi)] \leq \frac{E[f(\xi)]}{\alpha} + \frac{E[g(\xi)]}{1 - \alpha}$$

for any $0 < \alpha < 1$.

Proof: For any $0 < \alpha < 1$, it follows from Definition 2.3 and Axiom 3 that

$$\begin{aligned}
E[f(\xi)+g(\xi)] &= \int_0^\infty \mathcal{M}\{f(\xi)+g(\xi)\geq x\}\mathrm{d}x \\
&\leq \int_0^\infty \mathcal{M}\{\{f(\xi)\geq \alpha x\}\bigcup\{g(\xi)\geq (1-\alpha)x\}\}\mathrm{d}x \\
&\leq \int_0^\infty \mathcal{M}\{f(\xi)\geq \alpha x\}\mathrm{d}x + \int_0^\infty \mathcal{M}\{g(\xi)\geq (1-\alpha)x\}\mathrm{d}x \\
&= \frac{E[f(\xi)]}{\alpha} + \frac{E[g(\xi)]}{1-\alpha}.
\end{aligned}$$

The proof is complete.

Let ξ, η be uncertain variables. In order to illustrate there have a connection that $E[\xi\eta] = E[\xi]E[\eta]$ between their expected values, we introduced the following example.

Example 3.2 Let ξ be a linear uncertain variable $\mathcal{L}(b, c)$ with uncertainty distribution Φ_1, η be a zigzag uncertain variable $\mathcal{L}(0, b, c)$ with uncertainty distribution Φ_2. Then the expected values are as follows,

$$E[\xi] = \frac{b+c}{2}, \quad E[\eta] = \frac{2b+c}{4}.$$

We obtain

$$E[\xi]E[\eta] = \frac{2b^2+3bc+c^2}{8}.$$

If $\alpha \geq 0.5$, the inverse uncertainty distributions are

$$\Phi_1^{-1}(\alpha) = (1-\alpha)b + \alpha c, \quad \Phi_2^{-1}(\alpha) = (2-2\alpha)b + (2\alpha-1)c.$$

According to Theorem 2.2 and Theorem 2.3, we know that

$$\begin{aligned}
E[\xi\eta] &= \int_{0.5}^1 \Phi_1^{-1}(\alpha)\Phi_2^{-1}(\alpha)\mathrm{d}\alpha \\
&= \int_{0.5}^1 [(1-\alpha)b+\alpha c][(2-2\alpha)b+(2\alpha-1)c] \\
&= \frac{2b^2-10bc+5c^2}{24}.
\end{aligned}$$

Thus, if $4b^2 + 19bc - 2c^2 = 0$, we can obtain $E[\xi\eta] = E[\xi]E[\eta]$.

Theorem 3.4 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain variables with finite expected values. Then we have*

$$V[\sum_{i=1}^n \xi_i] = \sum_{i=1}^n V[\xi_i],$$

if $E[\xi_i\xi_j] = E[\xi_i]E[\xi_j]$.

Proof: It follows from the definition of variance and Theorem 2.3 that

$$\begin{aligned} V[\sum_{i=1}^{n}\xi_i] &= E[(\xi_1+\xi_2+\cdots+\xi_n-E[\xi_1]-E[\xi_2]-\cdots-E[\xi_n])^2] \\ &= \sum_{i=1}^{n}E[(\xi_i-E[\xi_i])^2]+\sum_{i\neq j}\sum_{i=1}^{n-1}E[(\xi_i-E[\xi_i])(\xi_j-E[\xi_j])]. \end{aligned}$$

Since $E[\xi_i\xi_j]=E[\xi_i]E[\xi_j]$, we get

$$\begin{aligned} E[(\xi_i-E[\xi_i])(\xi_j-E[\xi_j])] &= E[\xi_i\xi_j-\xi_iE[\xi_j]-\xi_jE[\xi_i]+E[\xi_i]E[\xi_j]] \\ &= E[\xi_i]E[\xi_j]-E[\xi_i]E[\xi_j]-E[\xi_j]E[\xi_i]+E[\xi_i]E[\xi_j] \\ &= 0. \end{aligned}$$

Thus

$$V[\sum_{i=1}^{n}\xi_i]=\sum_{i=1}^{n}E[(\xi_i-E[\xi_i])^2]=\sum_{i=1}^{n}V[\xi_i].$$

The theorem is proved.

Theorem 3.5 *Let $\xi_1,\xi_2,\cdots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. If $\xi_1,\xi_2,\cdots,\xi_n$ have finite expected values, then*

(1) $E[\bigvee_{i=1}^{n}(a_i\xi_i)]=\bigvee_{i=1}^{n}a_iE[\xi_i]$.

(2) $E[\bigwedge_{i=1}^{n}(a_i\xi_i)]=\bigwedge_{i=1}^{n}a_iE[\xi_i]$.

Proof: (1) Take $\xi=\bigvee_{i=1}^{n}(a_i\xi_i)$ and let Ψ be the uncertainty distribution of ξ. According to Definition 2.3 and Theorem 2.2, we have

$$\Psi^{-1}=(a_1\Phi_1^{-1}(\alpha))\bigvee(a_2\Phi_2^{-1}(\alpha))\bigvee\cdots\bigvee(a_n\Phi_n^{-1}(\alpha)).$$

It follows from Theorem 2.3 that

$$
\begin{aligned}
E[\bigvee_{i=1}^{n}(a_i\xi_i)] &= \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha \\
&= \int_0^1 (a_1\Phi_1^{-1}(\alpha))\bigvee(a_2\Phi_2^{-1}(\alpha))\bigvee\cdots\bigvee(a_n\Phi_n^{-1}(\alpha))\mathrm{d}\alpha \\
&= \int_0^1 \max_{1\le i\le n}(a_i\Phi_i^{-1}(\alpha))\mathrm{d}\alpha \\
&= \max_{1\le i\le n}\int_0^1 a_i\Phi_i^{-1}(\alpha)\mathrm{d}\alpha \\
&= \bigvee_{i=1}^{n}\int_0^1 a_i\Phi_i^{-1}(\alpha)\mathrm{d}\alpha \\
&= \bigvee_{i=1}^{n} a_iE[\xi_i].
\end{aligned}
$$

(2) In a similar proof, we can obtain $E[\bigwedge_{i=1}^{n}(a_i\xi_i)] = \bigwedge_{i=1}^{n} a_iE[\xi_i]$.

Theorem 3.6 *Let ξ and η be independent uncertain variables with finite expected values. Then*

$$E[|\xi-\eta|] = |E[\xi]-E[\eta]|.$$

Proof: If $\xi\ge\eta$, it follows from Remark 2.1 that

$$E[|\xi-\eta|] = E[\xi-\eta] = E[\xi+(-\eta)] = E[\xi]-E[\eta] = |E[\xi]-E[\eta]|.$$

According to Theorem 3.2, we have $E[\xi]-E[\eta]\ge 0$.
If $\xi<\eta$,

$$E[|\xi-\eta|] = E[\eta-\xi] = E[\eta]-E[\xi] = |E[\xi]-E[\eta]|.$$

The theorem is verified.

4 Conclusions

In the setting of uncertainty theory, some theorems and properties of expected value are introduced in this paper. We investigated the properties of expected value such as monotonicity, constant preserving, and so on. In addition, the definition of sublinear expectation was given, which is a special case of expectation.

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