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Two generalized (2+1)-dimensional hierarchies and Darboux transformations

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Abstract

Based on the TAH scheme, we construct the generalized (2+1)-dimensional S-mKdv hierarchy and the generalized (2+1)-dimensional Levi hierarchy, and we also generate their Hamiltonian structures. At last, we also obtain the Darboux transformations of the generalized (2+1)-dimensional Levi hierarchy.

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1 Introduction

Based on Lax pairs, a large number of (1+1)-dimensional integrable systems have been obtained [1-3]. Tu Guizhang et al.[4] presented a new method for generating (2+1)-dimensional hierarchies of evolution equations, which was called TAH scheme. The main idea of TAH scheme as follows [4].

Let $\mathcal{A}$ be an associative algebra over the field $\mathcal{K} = \mathcal{R}$ or $\mathcal{C}$. We introduce a residue operator on an associative algebra $\mathcal{A}[\xi]$ which consists of all pseudodifferential operators $\sum_{i=-\infty}^{N} a_i\xi^i$, where ξ stands for an operator defined by

$$\xi f = f\xi + (\partial_y f), \quad f \in \mathcal{A}. \tag{1}$$

By repeatedly applying the above formula, that will get

$$\xi^n f = \sum_{i\geq 0} \begin{pmatrix} n \\ i \end{pmatrix} (\partial^i f)\xi^{n-i}, n \in \mathcal{Z}, \tag{2}$$

where $\mathcal{Z}$ is the set of integers, and $\begin{pmatrix} n \\ i \end{pmatrix} = \frac{n(n-1)\cdots(n-i+1)}{i!}, \quad i \geq 0, \quad n \in \mathcal{Z}$.

Then, fix a matrix operator $U = U(\lambda + \xi, u) \in \mathcal{A}[\xi]$ which depends on a parameter λ and a vector function $u = (u_1, \cdots, u_p)^T$. Solving the equation $V_x = [U, V]$, where $V = \sum V_n \lambda^{-n}$. By solving the recursion relation among $g^{(n)} = (g_1^{(n)}, \cdots, g_p^{(n)})$, where $g_i^{(n)}$ comes from the expansion $\langle V, \frac{\partial U}{\partial u_i} \rangle = \sum_n g_i^{(n)} \lambda^{-n}$, where $\langle a, b \rangle = tr(R(ab)), \quad a, b \in \mathcal{A}[\xi]$.

Next, we try to find an operator J and form the hierarchy $u_{t_n} = Jg^{(n)}$. At last, by using the trace identity $\frac{\delta}{\delta U_i} \langle V, U_\lambda \rangle = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \langle V, \frac{\partial U}{\partial u_i} \rangle, \ i = 1, 2, \cdots, p$, the Hamiltonian structure of the above equation will be obtained.

Searching for Darboux transformations of soliton equations becomes more and more meaningful. There are some ways for generating Darboux transformations of soliton equations by starting from isospectral problems [5,6].

2 The generalized (2+1)-dimensional S-mKdv hierarchy and its Hamiltonian structure

We consider the isospectral problems

$$\begin{cases} \varphi_x = U\varphi, \quad U = \begin{pmatrix} \lambda + \xi & q + r \\ q - r & -(\lambda + \xi) \end{pmatrix}, \\ \varphi_t = V\varphi, \quad V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \sum_{m \geq 0} \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix} \lambda^{-m}. \end{cases} \tag{3}$$

Solving the stationary matrix $V_x = [U, V]$ gives rise to

$$\begin{cases} A_{nx} = A_{ny} + (q + r)C_n - B_n(q - r), \\ B_{nx} = 2B_{n+1} + 2B_n\xi + B_{ny} + (q + r)D_n - A_n(q + r), \\ C_{nx} = -2C_{n+1} - 2C_n\xi - C_{ny} + (q - r)A_n - D_n(q - r), \\ D_{nx} = -D_{ny} + (q - r)B_n - C_n(q + r), \\ B_0 = (q + r)\xi^{-1}, \ C_0 = (q - r)\xi^{-1}, \\ \partial_- A_0 = (q + r)(q - r)_y \xi^{-2} + O(\xi^{-3}), \\ \partial_+ D_0 = (q - r)(q + r)_y \xi^{-2} + O(\xi^{-3}). \end{cases} \tag{4}$$

By using (1), (2) and from (4), we can get

$$A_1 = -\frac{1}{2}(q + r)(q - r)\xi^{-1}, \quad D_1 = \frac{1}{2}(q + r)(q - r)\xi^{-1},$$

$$B_1 = \frac{1}{2}\{[(q + r)_x - (q + r)_y - (q + r)^2(q - r)]\xi^{-1} - 2(q + r) + O(\xi^{-2})\},$$

$$C_1 = \frac{1}{2}\{[-(q - r)_x - (q - r)_y - (q - r)^2(q + r)]\xi^{-1} - 2(q - r) + O(\xi^{-2})\}.$$

Based on (3) and the TAH scheme, we are easy to have the following (2+1)-dimensional hierarchy of evolution equations

$$u_{t_n}=\begin{pmatrix} q\\ r\end{pmatrix}_{t_n}=\begin{pmatrix} 2R(B_{n+1})\\ -2R(C_{n+1})\end{pmatrix}=J_1\begin{pmatrix} R(B_{n+1}+C_{n+1})\\ R(-B_{n+1}+C_{n+1})\end{pmatrix}, \qquad (5)$$

where $J_1=\begin{pmatrix} 1 & -1\\ -1 & -1\end{pmatrix}$.

When $n=1$, the hierarchy (5) can be written as

$$u_{t_1}=\begin{pmatrix} q\\ r\end{pmatrix}_{t_1}=\begin{pmatrix} 2R(B_2)\\ -2R(C_2)\end{pmatrix}=\begin{pmatrix} \frac{1}{2}\partial_-[(q+r)_x-(q+r)_y-(q+r)^2(q-r)]\\ -\frac{1}{2}\partial_+[\frac{1}{2}(q-r)_x+(q-r)_y+(q-r)^2(q+r)]\end{pmatrix}.$$

This is the generalized (2+1)-dimensional Schrödinger equation.

Next, we need to consider the spectral matrices U, V in (3). Therefore, we have

$$<V,\frac{\partial U}{\partial q}>=R(B+C), \quad <V,\frac{\partial U}{\partial r}>=R(-B+C), \quad <V,\frac{\partial U}{\partial \lambda}>=R(A-D).$$

Substituting the above results into the trace identity, we have

$$\begin{pmatrix} R(B_n+C_n)\\ R(-B_n+C_n)\end{pmatrix}=\frac{\delta}{\delta u}(\frac{D_{n+1}-A_{n+1}}{n})=\frac{\delta H_n^{(1)}}{\delta u}, \qquad H_n^{(1)}=\frac{D_{n+1}-A_{n+1}}{n}.$$

So the above S-mKdv hierarchy (5) has the following Hamiltonian form

$$u_{t_n}=J_1\begin{pmatrix} R(B_{n+1}+C_{n+1})\\ R(-B_{n+1}+C_{n+1})\end{pmatrix}=J_1\frac{\delta H_{n+1}^{(1)}}{\delta u}.$$

3 The generalized (2+1)-dimensional Levi hierarchy and Dardoux transformations

3.1 The generalized (2+1)-dimensional Levi hierarchy

We consider the following Lax matrices

$$U=\begin{pmatrix} 0 & -q\\ -1 & (\lambda+\xi)-r\end{pmatrix}, \quad V=\begin{pmatrix} A & B\\ C & D\end{pmatrix}=\sum_{m\geq 0}\begin{pmatrix} A_m & B_m\\ C_m & D_m\end{pmatrix}\lambda^{-m}. \qquad (6)$$

Substituting the above matrices U and V into the equation $V_x=[U,V]$, we find that

$$\begin{cases} A_{nx}=B_n-qC_n,\\ B_{nx}=-B_{n+1}-B_n\xi-qD_n+A_nq+B_nr,\\ C_{nx}=C_{n+1}+\xi C_n-A_n-rC_n+D_n,\\ D_{nx}=D_{ny}-B_n-rD_n+D_nr+C_nq,\\ B_0=C_0=0,\ A_0=\xi^{-1},\ D_0=0,\ B_1=\xi^{-1}q,\\ C_1=\xi^{-1},\ A_1=-\partial^{-1}q_y\xi^{-2}+O(\xi^{-3}),\ D_1=0.\end{cases} \qquad (7)$$

Note $V_+^{(n)} = \sum_{m=0}^{n}(A_m e_1(0) + D_m e_2(0) + B_m e_3(0) + C_m e_4(0))\lambda^{n-m} = \lambda^n V - V_-^{(n)}$, we have by tedious computation that $-V_{+x}^{(n)} + [U, V^{(n)}] = B_{n+1}e_3(0) - C_{n+1}e_4(0)$.

Set $V^{(n)} = V_+^{(n)} - C_{n+1}e_2(0)$, one infers that

$$-V_{+x}^{(n)} + [U, V^{(n)}] = C_{n+1,x}e_2(0) + (C_{n+1}q - B_{n+1})e_3(0).$$

So we have the generalized (2+1)-dimensional Levi hierarchy

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \begin{pmatrix} -R(D_{n+1,x}) \\ -R(C_{n+1,x}) \end{pmatrix} = J_2 \begin{pmatrix} -R(C_{n+1}) \\ -R(D_{n+1}) \end{pmatrix}, \tag{8}$$

where $J_2 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}$.

Next, we need to consider the spectral matrices U, V in (6). Therefore, we have

$$< V, \frac{\partial U}{\partial q} >= R(-C), \quad < V, \frac{\partial U}{\partial r} >= R(-D), \quad < V, \frac{\partial U}{\partial \lambda} >= R(D).$$

Substituting the above results into the trace identity, we are easy to get

$$\begin{pmatrix} R(-C_n) \\ R(-D_n) \end{pmatrix} = -\frac{\delta}{\delta u}(\frac{D_{n+1}}{n}) = \frac{\delta H_n^{(2)}}{\delta u}, \quad H_n^{(2)} = -\frac{D_{n+1}}{n}.$$

The above Levi hierarchy (8) can be written as the Hamiltonian form

$$u_{t_n} = J_2 \begin{pmatrix} -R(C_{n+1}) \\ -R(D_{n+1}) \end{pmatrix} = J_2 \frac{\delta H_{n+1}^{(2)}}{\delta u}.$$

3.2 The Dardoux transformations of (9)

Let $n = 2$, the hierarchy (8) reduces to a new equation as follows

$$\begin{cases} q_{t_2} = -\partial_x \partial_-^{-1}(-q_{xx} + 2q_{xy} + 2q_x r + 2qr_x + 2qr_y - 2q_y r - 2\partial^{-1}q_y q), \\ r_{t_2} = -(r_x + r_y + r^2 - 2q)_x, \end{cases} \tag{9}$$

whose Lax pair matrices present that

$$\begin{cases} \varphi_x = U_1 \varphi, \\ \varphi_y = U_2 \varphi, \\ \varphi_t = V \varphi, \end{cases} \tag{10}$$

where

$$U_1 = \begin{pmatrix} 0 & -q \\ -1 & \lambda - r \end{pmatrix}, U_2 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}, V = \begin{pmatrix} \lambda^2 - q + \partial^{-1}q_y & q\lambda - q_x + q_y + qr \\ \lambda + r & q \end{pmatrix}.$$

Based on the spectral problem (10), we consider the Darboux transformation [7] $\varphi' = T\varphi$, and require φ' satisfying the spectral problem

$$\begin{cases} \varphi'_x = U'_1\varphi', \\ \varphi'_y = U'_2\varphi', \\ \varphi'_t = V'\varphi', \end{cases} \tag{11}$$

where T is a 2×2 matrix, U'_1, U'_2, V' have the same forms as U_1, U_2, V expect replacing q, r by q', r'. It is easy to see that T meets $T_x + TU_1 = U'_1T$, $T_y + TU_2 = U'_2T$, $T_t + TV = V'T$.

Assume that $\phi = (\phi_1, \phi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are two fundamental solutions of the spectral problem (10), so one defines the matrix [7,8]

$$T = T(\lambda) = \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} \begin{pmatrix} A_0 & B_0 \\ -\delta & \delta(\lambda + D_0) \end{pmatrix} = \begin{pmatrix} \delta A_0 & \delta B_0 \\ -1 & \lambda + D_0 \end{pmatrix},$$

where $\delta B_0 = -\delta A_0 \frac{\phi_1}{\phi_2}$, $D_0 = \frac{\phi_1}{\phi_2} - \lambda_1$. When $\delta A_0 = 1$, then $\delta B_0 = -\frac{\phi_1}{\phi_2}$.

Next, we set the matrix U'_1 decided by (11) has the same form as U_1, where $U'_1 = \begin{pmatrix} 0 & -q' \\ -1 & \lambda - r' \end{pmatrix}$, and wish to find the relations of the potentials q, r and q', r'.

So, we need to set $T^{-1} = T^*/\mathrm{detT}$,

$$(T_x + TU_1)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}. \tag{12}$$

It is easy to see that $f_{11}(\lambda)$, $f_{12}(\lambda)$, $f_{21}(\lambda)$, $f_{22}(\lambda)$ are second-order polynomials on λ. So (12) can be written as

$$T_x + TU_1 = P(\lambda)T, \tag{13}$$

where $P(\lambda) = \begin{pmatrix} 0 & p_{12}^{(0)} \\ -1 & \lambda + p_{22}^{(0)} \end{pmatrix}$.

From (13), we have

$$\begin{cases} \delta_x A_0 + \delta A_{0x} - \delta B_0 = -p_{12}^{(0)}, \\ \delta_x B_0 + \delta B_{0x} - \delta A_0 q + \delta B_0\lambda - \delta B_0 r = (\lambda + D_0)p_{12}^{(0)}, \\ -\lambda - D_0 = -\delta A_0 - \lambda - p_{22}^{(0)}, \\ D_{0x} + q + (\lambda + D_0)(\lambda - r) = -\delta B_0 + (\lambda + D_0)(\lambda + p_{22}^{(0)}). \end{cases} \tag{14}$$

Comparing the coefficients of $\lambda^j (j = 0, 1)$ in (14), we have the following relations

$$\begin{cases} p_{12}^{(0)}=\delta B_0=-q', \\ p_{22}^{(0)}=-r=-r', \\ \delta_x A_0+\delta A_{0x}=0, \quad \text{choose} \quad \delta \mathrm{A}_0=1, \\ (\delta B_0)_x-q-\delta B_0 r=\delta B_0 D_0, \\ r+D_0=1, \\ D_{0x}+q=-\delta B_0, \\ D_0=-\delta B_0. \end{cases} \tag{15}$$

Next we set the matrix V' decided by (11) has the same form as V, where $V'=\begin{pmatrix} \lambda^2-q'+\partial^{-1}q'_y & q'\lambda-q'_x+q'_y+q'r' \\ \lambda+r' & q' \end{pmatrix}$, and we wish to find the relations of the potentials q, r and q', r'.

We note $T^{-1}=T^*/\mathrm{detT}$,

$$(T_y+TV)T^*=\begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}. \tag{16}$$

It is easy to see that $g_{11}(\lambda),\ g_{12}(\lambda),\ g_{21}(\lambda),\ g_{22}(\lambda)$ are second-order polynomials on λ. So (16) can be written as

$$T_t+TV=Q(\lambda)T, \tag{17}$$

where $Q(\lambda)=\begin{pmatrix} \lambda^2+q_{11}^{(0)} & q_{12}^{(1)}\lambda+q_{12}^{(0)} \\ \lambda+q_{21}^{(0)} & q_{22}^{(0)} \end{pmatrix}$.

Solving (17), we have

$$(\delta A_0)_y+\delta A_0\lambda^2+\delta A_0(-q+\partial^{-1}q_y)+\delta B_0(\lambda+r)=\delta A_0\lambda^2+\delta A_0 q_{11}^{(0)}-q_{12}^{(0)}\lambda-q_{12}^{(0)}, \tag{18}$$

$$(\delta B_0)_y+\delta A_0(q\lambda-q_x+q_y+qr)+\delta B_0 q=\delta B_0(\lambda^2+q_{11}^{(0)})+(\lambda+D_0)(q_{12}^{(1)}\lambda+q_{12}^{(0)}), \tag{19}$$

$$-\lambda^2-(-q+\partial^{-1}q_y)+(\lambda+r)(\lambda+D_0)=\delta A_0(\lambda+q_{21}^{(0)})-q_{22}^{(0)}, \tag{20}$$

$$D_{0y}-q\lambda-(-q_x+q_y+qr)+(\lambda+D_0)q=\delta B_0(\lambda+q_{21}^{(0)})+(\lambda+D_0)q_{22}^{(0)}. \tag{21}$$

Comparing the coefficients of $\lambda^j (j=0,1,2)$ in (18-21), we have

$$\begin{cases} q_{12}^{(1)}=-\delta B_0=q', \\ r+D_0=\delta A_0, \\ q_{22}^{(0)}=-\delta B_0=q', \\ q_{12}^{(0)}=\delta A_0 q+\delta B_0 D_0, \\ q_{11}^{(0)}=\partial^{-1}q_y+\delta B_0, \\ q_{21}^{(0)}=-\partial^{-1}q_y+q+D_0 r-\delta B_0=r', \\ -q_{11}^{(0)}+q_{12}^{(0)}=q_{21}^{(0)}-q_{22}^{(0)}. \end{cases} \tag{22}$$

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