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Super-biderivations of the super Virasoro algebra

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Abstract

In this paper we investigate super-biderivations of the super Virasoro algebra. The super Virasoro algebra is a Lie superalgebra equipped with a basis $\{L_m, I_m, G_m \mid m \in Z\}$ and nontrivial Lie super-brackets: $[L_m, L_n] = (n-m)L_{m+n}$, $[L_m, I_n] = nI_{m+n}$, $[L_m, G_n] = (n-m)G_{m+n}$, $[I_m, G_n] = G_{m+n}$. Finally, we prove that all skew-supersymmetric super-biderivations of the super Virasoro algebra are inner super-biderivations.

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1 Introduction

Recently, many authors have investigated biderivations of both Lie algebras and Lie superalgebras. In [3,7], the authors proved that each skew-symmetric biderivation on the Schrödinger-Virasoro algebra and a simple generalized Witt algebra over a field of characteristic 0 is a inner biderivation. Later on, super-biderivations of many Lie superalgebras were studied in [4,9,10]. In [5], the author introduced the concept of the super Virasoro algebra. In [11], the

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authors studied Lie super-bialgebra and quantization of the super Virasoro algebra.

The super Virasoro algebra S is a Lie superalgebra whose even part $S_{\bar{0}}$ has a basis $\{L_m, I_m \mid m \in Z\}$ and odd part $S_{\bar{1}}$ has a basis $\{G_m \mid m \in Z\}$, equipped with the following nontrivial Lie super-brackets ($m, n \in Z$): $[L_m, L_n] = (n - m)L_{m+n}$, $[L_m, I_n] = nI_{m+n}$, $[L_m, G_n] = (n-m)G_{m+n}$, $[I_m, G_n] = G_{m+n}$.

Obviously, we know that S contains many important subalgebras. For example,

- $W = \oplus_{m\in Z} L_m$ is in fact the well-known centerless Virasoro algebra.
- $H = (\oplus_{m\in Z} L_m)\oplus(\oplus_{m\in Z} I_m)$ is the centerless twisted Heisenberg-Virasoro algebra.

Note that S is Z-graded: $S = \underset{m\in Z}{\oplus} S_m$, $S_m = \text{span}\{L_m, I_m, G_m\}$.

Here is a detailed outline of the contents of the main parts of the article. In Section 2, we review some conclusions about super-biderivations of Lie superalgebras. In Section 3, we prove that all skew-supersymmetric super-biderivations of the super Virasoro algebras are inner super-biderivations.

For the readers' convenience, we give some notations used in this paper. Denote by C, Z the sets of complex numbers, integers. We assume that all vector spaces are based on C and the degree of x or ϕ is denoted by $|x|$ or $|\phi|$. In addition, x is always assumed to be homogeneous when $|x|$ occurs. Denote by $hg(L)$ the set of all homogeneous elements of L where L be a superspace.

2 Preliminaries

In this section, we shall summarize some basic concepts about super-bideivations of Lie superalgebras in [4,9].

Definition 2.1 We call a bilinear map ϕ:$L \times L \longrightarrow L$ a super-biderivation of L if it satisfies the following two equations ($x, y, z \in hg(L)$):

$$\phi([x, y], z) = (-1)^{|\phi||x|}[x, \phi(y, z)] + (-1)^{|y||z|}[\phi(x, z), y], \tag{1}$$

$$\phi(x, [y, z]) = [\phi(x, y), z] + (-1)^{(|\phi|+|x|)|y|}[y, \phi(x, z)]. \tag{2}$$

Proposition 2.2 We say that a super-biderivation ϕ of L is a skew-supersymmetric super-biderivation if ϕ satisfies the following condition ($x, y \in hg(L)$):

$$skew - supersymmetry: \quad \phi(x, y) = -(-1)^{|x||y|}\phi(y, x). \tag{3}$$

Definition 2.3 A super-biderivation ϕ of homogenous $\gamma \in Z_2$ of L is a super-biderivation such that $\phi(L_\alpha, L_\beta) \subseteq L_{\alpha+\beta+\gamma}$ for any $\alpha, \beta \in Z_2$. Denote by $\mathrm{BDer}_\gamma(L)$ the set of all super-biderivations of homogenous γ of L. Obviously, $\mathrm{BDer}\,(L) = \mathrm{BDer}\,_{\bar{0}}(L) \oplus \mathrm{BDer}\,_{\bar{1}}(L)$.

Lemma 2.4 If the map $\phi_\lambda{:}L \times L \longrightarrow L$, defined by $\phi_\lambda(x, y) = \lambda[x, y]$ for any $x, y \in hg(L)$, where $\lambda \in C$, then ϕ_λ is a skew-supersymmetric super-biderivation of L. We call this class super-biderivations inner super-biderivations.

Lemma 2.5 Let ϕ be a skew-supersymmetric super-biderivation on L, then we get $[\phi(x, y), [u, v]] = (-1)^{|\phi|(|x|+|y|)}[[x, y], \phi(u, v)]$ for any $x, y, u, v \in hg(L)$.

Lemma 2.6 Let ϕ be a skew-supersymmetric super-biderivation on L. If $|x| + |y| = \bar{0}$, then $[\phi(x, y), [x, y]] = 0$ for any $x, y \in hg(L)$.

Lemma 2.7 Let ϕ be a skew-supersymmetric super-biderivation on L. If $[x, y] = 0$, then $\phi(x, y) \in Z([L, L])$, where $Z([L, L])$ is the center of $[L, L]$.

3 Super-biderivations of the super Virasoro algebra

In this section, we would like to compute super-biderivations of the super Virasoro algebras.

Lemma 3.1 Every super-skewsymmetric super-biderivation on the super Virasoro algebra S is an inner super-biderivation.

Proof Suppose ϕ is a super-biderivation of the super Virasoro algebra S. Assume that $\phi(L_0, L_n) = \Sigma_{m\in Z}(a^n_m L_m + b^n_m I_m + c^n_m G_m)$, $\phi(L_0, I_n) = \Sigma_{m\in Z}(d^n_m L_m + e^n_m I_m + f^n_m G_m)$, $\phi(L_0, G_n) = \Sigma_{m\in Z}(g^n_m L_m + h^n_m I_m + \triangle^n_m G_m)$, where a^n_m, b^n_m, c^n_m, d^n_m, e^n_m, f^n_m, g^n_m, h^n_m, $\triangle^n_m \in C$ for any $m, n \in Z$.

According to Lemma 2.7, then $\phi(L_0, L_0), \phi(L_0, I_0), \phi(L_0, G_0) \in Z([S, S])$. Hence, $\phi(L_0, L_0) = \phi(L_0, I_0) = \phi(L_0, G_0) = 0$.

Due to $L_m \in S_{\bar{0}}$, then $|L_m| + |L_n| = \bar{0}$ for any $m, n \in Z$. By Lemma 2.6, then we obtain

$$[[L_0, L_n], \phi(L_0, L_n)] = 0.$$

Furthermore,

$$n[L_n, \Sigma_{m\in Z}(a^n_m L_m + b^n_m I_m + c^n_m G_m)] = 0.$$

One has

$$a^n_m(m - n) = b^n_m m = c^n_m(m - n) = 0.$$

Thus, $a_m^n = c_m^n = 0$ for $m \neq n$ and $b_m^n = 0$ for $m \neq 0$. So we get $\phi(L_0, L_n) = a_n^n L_n + b_0^n I_0 + c_n^n G_n$.

By Lemma 2.5, we have

$$[\phi(L_0, L_n), [L_0, L_1]] = (-1)^{|\phi|(|L_0|+|L_n|)}[[L_0, L_n], \phi(L_0, L_1)].$$

Hence, we deduce that $a_n^n = na_1^1$ and $c_n^n = nc_1^1$.

Let $\lambda = a_1^1$, $\mu = c_1^1$, then we have

$$\phi(L_0, L_n) = \lambda n L_n + b_0^n I_0 + \mu n G_n.$$

By Lemma 2.5, we have

$$[\phi(L_0, L_k), [L_0, I_n]] = (-1)^{|\phi|(|L_0|+|I_n|)}[[L_0, L_k], \phi(L_0, I_n)]$$

.

One deduces that

$$\phi(L_0, L_n) = \lambda n L_n + b_0^n I_0,$$

$$\phi(L_0, I_n) = \lambda n G_n + d_0^n I_0.$$

Set $x = L_0, y = I_0, z = G_n$ in (2), then

$$\phi(L_0, [I_0, G_n]) = [\phi(L_0, I_0), G_n] + [I_0, \phi(L_0, G_n)].$$

Hence, we have $\phi(L_0, G_n) = \Sigma_{m \in Z}(\triangle_m^n G_m)$.

Set $x = L_0, y = L_0, z = G_n$ in (2), then

$$\phi(L_0, [L_0, G_n]) = [\phi(L_0, L_0), G_n] + [L_0, \phi(L_0, G_n)].$$

Hence, we have $\phi(L_0, G_n) = \triangle_n^n G_n$.

Set $x = L_0, y = L_k, z = G_n$ in (2), then we have

$$\phi(L_0, [L_k, G_n]) = [\phi(L_0, L_k), G_n] + [L_k, \phi(L_0, G_n)],$$

This shows that $\triangle_n^n = \lambda n$ and $b_0^n = 0$.

Set $x = L_0, y = I_k, z = G_n$ in (2), then we have

$$\phi(L_0, [I_k, G_n]) = [\phi(L_0, I_k), G_n] + [I_k, \phi(L_0, G_n)],$$

Therefore, we have $d_0^n = 0$.

Finally, we have proved the following equations ($n \in Z$):

$$\phi(L_0, L_n) = \lambda[L_0, L_n],$$

$$\phi(L_0, I_n) = \lambda[L_0, I_n],$$

$$\phi(L_0, G_n) = \lambda[L_0, G_n].$$

For any $z \in S$, we get

$$\phi(L_0, z) = \lambda[L_0, z].$$

Due to $|\phi| = \bar{0}$, and according to Lemma 2.5, we obtain

$$[\phi(x, y), [L_0, z]] = [[x, y], \phi(L_0, z)].$$

Furthermore we have $[\phi(x, y) - \lambda[x, y], [L_0, z]] = 0$. According to the arbitrary of z, then $\phi(x, y) - \lambda[x, y] = 0$.

Thus, $\phi(x, y) = \lambda[x, y]$.

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