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Regularity for Minimizers of Some Anisotropic Integral Functionals with Nonstandard Growth

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Abstract

This paper deals with anisotropic integral functionals of the type

$$\mathcal{I}(u)=\int_\Omega f(x,Du(x))dx,$$

where the Carathéodory function $f(x,z):\Omega\times \mathrm{R}^n\to \mathrm{R}$ satisfies the growth condition

$$\mu\sum_{i=1}^n |z_i|^{p_i}-g(x)\le f(x,z)$$

for almost every $x\in\ \Omega$ and all $z\in \mathrm{R}^n$. We consider a minimizer $u:\Omega\in \mathrm{R}^n\to \mathrm{R}$ among all functions that agree on the boundary $\partial\Omega$ with some fixed boundary value u_* and with gradient constraints. We assume that the boundary datum u_* make the density $f(x,Du_*(x))$ more integrable and we prove that the minimizer u enjoys higher integrability.

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1 Introduction and statementa of result.

Let Ω be a bounded open subset in R^n, $n\ge 2$. For $p_1,\cdots,p_n\in(1,+\infty)$, we set

$$\bar p=\left(\frac{1}{n}\sum_{i=1}^n\frac{1}{p_i}\right)^{-1},\quad p_i'=\frac{p_i}{p_i-1},\quad p_{\max}=\max_{1\le i\le n}p_i$$

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be the harmonic mean of $p_1, \cdots, p_n$, the Hölder conjugate of p_i, and the maximum value of $p_1, \cdots, p_n$, respectively. In this paper we assume $\bar{p} < n$ and we denote $\bar{p}^*$ to be the Sobolev conjugate of $\bar{p}$, that is, $\bar{p}^* = \frac{n\bar{p}}{n-\bar{p}}$.

Let $T_k : \mathrm{R} \to \mathrm{R}$ be the truncation function of level $k > 0$,

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k; \\ k \cdot \operatorname{sign}(s), & \text{if } |s| > k. \end{cases}$$

We work in weak Lebesgue spaces $L^{\sigma}_{weak}(\Omega)$, known also as Marcinkiewicz spaces or Lorentz spaces $L^{\sigma,\infty}(\Omega)$: If $\sigma > 1$, then the space $L^{\sigma}_{weak}(\Omega)$ consists of all measurable functions $h(x)$ on Ω such that

$$\sup_{t>0} t|\{x \in \Omega : |h(x)| > t\}|^{\frac{1}{\sigma}} < \infty.$$

This condition is equivalently stated as

$$|||h|||_{\sigma} = \sup_{E\subset\Omega,|E|>0} \frac{1}{|E|^{\frac{1}{\sigma'}}} \int_E |h(x)|dx < \infty. \tag{1.1}$$

We let $\varphi : \Omega \to \mathrm{R}$ be a nonnegative function and $u_* \in W^{1,(p_i)}(\Omega)$ be such that $|Du_*(x)| \le \varphi(x)$, for a.e. $x \in \Omega$. We define

$$\mathcal{C}_{u_*,\varphi} = \{v \in u_* + W^{1,(p_i)}_0(\Omega) : |Dv(x)| \le \varphi(x), \text{ a.e. } x \in \Omega\}.$$

It is obvious that $Du_*(x) \in \mathcal{C}_{u_*,\varphi}$, thus $\mathcal{C}_{u_*,\varphi} \neq \emptyset$. It is easy to see that the set $\mathcal{C}_{u_*,\varphi}$ is convex and closed in $W^{1,(p_i)}(\Omega)$.

We consider anisotropic integral functionals of the type

$$\mathcal{I}(u) = \int_\Omega f(x, Du(x))dx, \tag{1.2}$$

where $u : \Omega \subset \mathrm{R}^n \to \mathrm{R}$ and $f : \Omega \times \mathrm{R}^n \to \mathrm{R}$ is a Carathéodory function, that is, $f(x, z)$ is measurable with respect to $x \in \Omega$ and continuous with resect to $z \in \mathrm{R}^n$. We assume that there exists a constant $\mu > 0$ and a function $g(x) : \Omega \to [0, +\infty)$ such that

$$\mu \sum_{i=1}^{n} |z_i|^{p_i} - g(x) \le f(x, z), \tag{1.3}$$

for almost every $x \in \Omega$ and for all $z \in \mathrm{R}^n$.

We consider the following problem:

$$u \in \mathcal{C}_{u_*,\varphi}, \tag{1.4}$$

$$\forall v \in \mathcal{C}_{u_*,\varphi}, \quad \int_\Omega f(x, Du(x))dx \le \int_\Omega f(x, Dv(x))dx. \tag{1.5}$$

The main result of this paper is the following theorem.

Theorem 1.1 *Assume that* $g(x), f(x, Du_*(x)) \in L^{\sigma}_{weak}(\Omega)$*,* $\sigma > 1$*. If* u *satisfies (1.4), (1.5) under (1.3), then*

(i) $\sigma < \frac{n}{\bar{p}} \Rightarrow u - u_* \in L^{\frac{n\bar{p}\sigma}{n-\bar{p}\sigma}}_{weak}(\Omega)$*;*

(ii) $\sigma = \frac{n}{\bar{p}} \Rightarrow \exists \alpha > 0 : e^{\alpha|u-u_*|} \in L^1(\Omega)$*;*

(iii) $\sigma > \frac{n}{\bar{p}} \Rightarrow u - u_* \in L^{\infty}(\Omega)$*.*

For some related regularity results of minimizers of integral functionals with nonstandard growth as well as nonlinear elliptic equations and systems, we refer the reader to [1-7].

Note that the condition on $g(x)$ and $f(x, Du_*(x))$ in Theorem 1 is slightly weaker than that of [1]. Note also that $\frac{n\bar{p}\sigma}{n-\bar{p}\sigma} > \frac{n\bar{p}}{n-\bar{p}}$.

2 Proof of Theorem 1.1.

In order to prove Theorem 1.1, we need a preliminary lemma, which can be found in [6, Proposition 2.2].

Lemma 2.1 *Let* $v \in W_0^{1,(p_i)}(\Omega)$*, and let* $M > 0, \gamma > 0$*, and* $k_0 \geq 0$*. Let for every* $k > k_0$*,*

$$\int_{\{|v|>k\}} \sum_{i=1}^{n} |D_i v|^{p_i} dx \leq M |\{|v| > k\}|^{\frac{\gamma \bar{p}}{\bar{p}^*}}$$

Then the following assertions hold:

(i) $\gamma < 1 \Rightarrow v \in L^{\frac{\bar{p}^*}{1-\gamma}}(\Omega)$*;*

(ii) $\gamma = 1 \Rightarrow \exists \alpha > 0 : e^{\alpha|v|} \in L^1(\Omega)$*;*

(iii) $\gamma > 1 \Rightarrow v \in L^{\infty}(\Omega)$*.*

Proof. For $u \in \mathcal{C}_{u_*,\varphi}$ and $k > 0$ we let

$$v = u_* + T_k(u - u_*) = \begin{cases} u_* + k, & u - u_* > k; \\ u, & |u - u_*| \leq k; \\ u_* - k, & u - u_* < -k. \end{cases}$$

It is obvious that

$$v \in u_* + W_0^{1,(p_i)}(\Omega) \tag{2.1}$$

and

$$Dv(x) = \begin{cases} Du_*(x), & |u - u_*| > k; \\ Du(x), & |u - u_*| \leq k. \end{cases} \tag{2.2}$$

It follows from $u, u_* \in \mathcal{C}_{u_*,\varphi}$ that $|Dv(x)| \leq \varphi(x)$ for a.e. $x \in \Omega$, which together with (2.1) implies $v \in \mathcal{C}_{u_*,\varphi}$. So we can use (1.5) with v as a test function that implies

$$\begin{aligned} &\int_{\{|u-u_*|\leq k\}} f(x, Du)dx + \int_{\{|u-u_*|>k\}} f(x, Du)dx = \int_{\Omega} f(x, Du)dx \\ \leq &\int_{\Omega} f(x, Dv)dx = \int_{\{|u-u_*|\leq k\}} f(x, Du)dx + \int_{\{|u-u_*|>k\}} f(x, Du_*)dx. \end{aligned} \tag{2.3}$$

Since u and u_* have finite energy, all the integrals above are finite, then we can drop $\int_{\{|u-u_*|\le k\}} f(x,Du)dx$ from both sides of (2.3) and we get

$$\int_{\{|u-u_*|>k\}} f(x,Du)dx \le \int_{\{|u-u_*|>k\}} f(x,Du_*)dx. \tag{2.4}$$

(1.3) together with (2.4) implies

$$\begin{aligned}
&\int_{\{|u-u_*|>k\}} \sum_{i=1}^n |D_i u - D_i u_*|^{p_i} dx \\
\le\ & 2^{p_{\max}-1}\left[\int_{\{|u-u_*|>k\}} \sum_{i=1}^n |D_i u|^{p_i} dx + \int_{\{|u-u_*|>k\}} \sum_{i=1}^n |D_i u_*|^{p_i} dx\right] \\
\le\ & \frac{2^{p_{\max}-1}}{\mu}\left[\int_{\{|u-u_*|>k\}} f(x,Du)dx + \int_{\{|u-u_*|>k\}} g(x)dx\right] \\
& +2^{p_{\max}-1}\int_{\{|u-u_*|>k\}} \sum_{i=1}^n |D_i u_*|^{p_i} dx. \\
\le\ & \frac{2^{p_{\max}}}{\mu}\left[\int_{\{|u-u_*|>k\}} f(x,Du_*)dx + \int_{\{|u-u_*|>k\}} g(x)dx\right] \\
=\ & \int_{\{|u-u_*|>k\}} h dx,
\end{aligned} \tag{2.5}$$

where

$$h(x) = \frac{2^{p_{\max}}}{\mu}\left[f(x,Du_*(x)) + g(x)\right] \in L^\sigma_{weak}(\Omega).$$

(1.1) implies

$$\int_{\{|u-u_*|>k\}} h dx \le |||h|||_{L^\sigma(\Omega)} |\{|u-u_*|>k\}|^{\frac{1}{\sigma'}} \tag{2.6}$$

Combining (2.5) with (2.6) we obtain

$$\int_{\{|u-u_*|>k\}} \sum_{i=1}^n |D_i u - D_i u_*|^{p_i} dx \le |||h|||_{L^\sigma(\Omega)} |\{|u-u_*|>k\}|^{\frac{1}{\sigma'}}.$$

Theorem 1.1 follows from Lemma 2.1.

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