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Two common fixed point theorems for a sequence of mappings in complete G - metric spaces

Valeriu Popa

"Vasile Alecsandri" University of Bacău
157 Calea Mărăşeşti, Bacău, 600115, ROMANIA

Alina-Mihaela Patriciu

"Dunărea de Jos" University of Galaţi
Faculty of Sciences and Environment
Department of Mathematics and Computer Sciences
111 Domnească Street, Galaţi, 800201, ROMANIA

Abstract. In this paper two common fixed point theorems for a sequence of mappings satisfying implicit relation in complete G - metric spaces are proved.

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1. Introduction

In [5], [6] Dhage introduced a new class of generalized metric spaces, named D - metric space. Mustafa and Sims [9], [10] proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced appropriate notion of generalized metric space, named G - metric space.

In fact, Mustafa, Sims and other authors [1] - [4], [7], [9] - [15] studied many fixed point results for self mappings in G - metric spaces under certain conditions. Quite recently, Meena and Nema [8] proved a common fixed point theorem for a sequence of mappings in complete G - metric spaces.

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [16], [17] and in other papers. Actually, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, ultra - metric spaces, convex metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, in two or three metric spaces, for single - valued mappings, hybrid pairs of mappings and multi - valued mappings. Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, partial metric spaces and ordered metric spaces.

With this method the proofs of some fixed point theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures.

The study of fixed points for mappings satisfying implicit relations in G - metric spaces is initiated in [18] - [23].

The study of fixed points for a sequence of mappings in G - metric spaces is recently initiated in [8].

2. Preliminaries

Definition 2.1 ([10]). *Let X be a nonempty set and $G : X^3 \to R_+$ be a function satisfying the following conditions:*

$(G_1) : G(x, y, z) = 0$ *if* $x = y = z$,

$(G_2) : 0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,

$(G_3) : G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$,

$(G_4) : G(x, y, z) = G(y, z, x) = G(z, x, y) = ...$ *(symmetry in all three variables),*

$(G_5) : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ *(rectangle inequality).*

Then the function G is called a G - metric on X and the pair (X, G) is called a G - metric space.

Remark 2.2 ([10]). *If $G(x, y, z) = 0$, then $x = y = z$.*

Definition 2.3 ([10]). *Let (X, G) be a G - metric space. A sequence $\{x_n\}$ in X is said to be*

a) G - convergent if for $\varepsilon > 0$, there exist $x \in X$ and $k \in N$ such that for all $m, n \geq k$, $G(x, x_n, x_m) < \varepsilon$,

b) G - Cauchy if for each $\varepsilon > 0$, there exists $k \in N$ such that for all $n, m, p \geq k$, $G(x_n, x_m, x_p) < \varepsilon$, that is $G(x_n, x_m, x_p) \to 0$ as $m, n, p \to \infty$.

A G - metric space (X, G) is said to be a complete G - metric space if every G - Cauchy sequence of (X, G) is G - convergent.

Lemma 2.4 ([10]). *Let (X, G) be a G - metric space. Then, the following properties are equivalent:*

1) $\{x_n\}$ is G - convergent to x;

2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$;

3) $G(x_n, x, x) \to 0$ as $n \to \infty$;

4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Lemma 2.5 ([10]). *If (X,G) is a G - metric space and $\{x_n\} \in X$, then the following properties are equivalent:*

1) the sequence $\{x_n\}$ is G - Cauchy,

2) For every $\varepsilon > 0$, there exists $k \in N$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq k$.

Lemma 2.6 ([10]). *$G(x,y,z)$ is jointly continuous in all three of its variables.*

Note that each G - metric on X generates a topology τ_G on X, whose base is a family of open G - balls $\{B_G(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_G(x,\varepsilon) = \{y \in X : G(x,y,y) < \varepsilon\}$ for all $x, y \in X$ and $\varepsilon > 0$.

A nonempty set $A \subset (X,G)$ is G - closed if $A = Cl\,(A)$.

Lemma 2.7 ([7]). *Let (X,G) be a G - metric space and A a subset of A. A is closed if for any G - convergent sequence $\{x_n\}$ in A with $\lim_{n\to\infty} x_n = x$, then $x \in A$.*

The purpose of this paper is to prove two general common fixed point theorems for a sequence of mappings satisfying a new type of implicit relation in G - metric spaces.

3. Implicit relations

Definition 3.1. *Let $\mathcal{F}_Q$ be the set of all continuous functions $F(t_1, ..., t_5) : R_+^5 \to R$ satisfying the following conditions*

(F_1) *F is nonincreasing in variables t_3, t_4, t_5,*

(F_2) *There exists a $h \in (0,1)$ such that for all $u, v \geq 0$, $F(u,v,u,v,u+v) \leq 0$ implies $u \leq hv$,*

(F_3) *$F(t,0,0,t,t) \leq 0$ implies $t = 0$.*

In the following examples, the proofs of property (F_1) is obviously.

Example 3.2. *$F(t_1, ..., t_5) = t_1 - at_2 - bt_3 - ct_4 - dt_5$, where $a, b, c, d \geq 0$ and $a + b + c + 2d < 1$.*

(F_2) : *Let $u, v \geq 0$ be and $F(u,v,u,v,u+v) = u - av - bu - cv - d\,(u+v) \leq 0$, which implies $u \leq hv$, where $0 \leq h = \frac{a+c+d}{1-(b+d)} < 1$.*

(F_3) : *$F(t,0,0,t,t) = t\,[1-(c+d)] \leq 0$ implies $t = 0$.*

Example 3.3. *$F(t_1, ..., t_5) = t_1 - k\max\{t_2, t_3, t_4, t_5\}$, where $k \in \left[0, \frac{1}{2}\right)$.*

(F_2) : *Let $u, v \geq 0$ be and $F(u,v,u,v,u+v) = u - k\,(u+v) \leq 0$, which implies $u \leq hv$ where $0 \leq h = \frac{k}{1-k} < 1$.*

(F_3) : *$F(t,0,0,t,t) = t(1-k) \leq 0$ implies $t = 0$.*

Example 3.4. *$F(t_1, ..., t_5) = t_1 - k\max\left\{t_2, t_3, \frac{t_4+t_5}{3}\right\}$, where $k \in [0,1)$.*

(F_2) : *Let $u, v \geq 0$ be and $F(u,v,u,v,u+v) = u - k\max\left\{u, v, \frac{u+2v}{3}\right\} \leq 0$. If $u > v$ then $u\,(1-k) \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$ where $0 \leq h = k < 1$.*

(F_3) : *$F(t,0,0,t,t) = t(1-k) \leq 0$ implies $t = 0$.*

Example 3.5. $F(t_1,...,t_5) = t_1^2 - at_2t_3 - bt_3t_4 - ct_4t_5$, *where* $a,b,c \geq 0$ *and* $0 < a+b+2c < 1$.

(F_2) : *Let* $u,v \geq 0$ *be and* $F(u,v,u,v,u+v) = u^2 - auv - buv - cv\,(u+v) \leq 0$. *If* $u > v$ *then* $u^2[1-(a+b+2c)] \leq 0$, *a contradiction. Hence* $u \leq v$, *which implies* $u \leq hv$ *where* $0 \leq h = \sqrt{a+b+2c} < 1$.

(F_3) : $F(t,0,0,t,t) = t^2(1-c) \leq 0$ *implies* $t = 0$.

Example 3.6. $F(t_1,...,t_5) = t_1 - at_2 - b\max\{2t_3, t_4+t_5\}$, *where* $a,b \geq 0$ *and* $a+3b < 1$.

(F_2) : *Let* $u,v \geq 0$ *be and* $F(u,v,u,v,u+v) = u - av - b\max\{2u, u+2v\} \leq 0$. *If* $u > v$ *then* $u[1-(a+3b)] \leq 0$, *a contradiction. Hence* $u \leq v$, *which implies* $u \leq hv$ *where* $0 \leq h = a+3b < 1$.

(F_3) : $F(t,0,0,t,t) = t(1-2b) \leq 0$ *implies* $t = 0$.

Example 3.7. $F(t_1,...,t_5) = t_1 - at_2 - b\max\{t_3+t_4, 2t_5\}$, *where* $a,b \geq 0$ *and* $a+4b < 1$.

(F_2) : *Let* $u,v \geq 0$ *be and* $F(u,v,u,v,u+v) = u - av - 2b\,(u+v) \leq 0$, *which implies* $u \leq hv$ *where* $0 \leq h = \frac{a+2b}{1-2b} < 1$.

(F_3) : $F(t,0,0,t,t) = t(1-2b) \leq 0$ *implies* $t = 0$.

Example 3.8. $F(t_1,...,t_5) = t_1^2 - at_2^2 - bt_3^2 - ct_4t_5$, *where* $a,b,c \geq 0$ *and* $a+b+2c < 1$.

(F_2) : *Let* $u,v \geq 0$ *be and* $F(u,v,u,v,u+v) = u^2 - av^2 - bu^2 - cv\,(u+v) \leq 0$. *If* $u > v$ *then* $u^2[1-(a+b+2c)] \leq 0$, *a contradiction. Hence* $u \leq v$, *which implies* $u \leq hv$ *where* $0 \leq h = \sqrt{a+b+2c} < 1$.

(F_3) : $F(t,0,0,t,t) = t^2(1-c) \leq 0$ *implies* $t = 0$.

Example 3.9. $F(t_1,...,t_5) = t_1 - a\max\{t_2,t_3\} - b\max\{t_4,t_5\}$, *where* $a,b \geq 0$ *and* $a+2b < 1$.

(F_2) : *Let* $u,v \geq 0$ *be and* $F(u,v,u,v,u+v) = u - a\max\{u,v\} - b\,(u+v) \leq 0$. *If* $u > v$ *then* $u[1-(a+2b)] \leq 0$, *a contradiction. Hence* $u \leq v$, *which implies* $u \leq hv$ *where* $0 \leq h = a+2b < 1$.

(F_3) : $F(t,0,0,t,t) = t(1-b) \leq 0$ *implies* $t = 0$.

4. Main results

Theorem 4.1. *Let* S *be a closed subset of a complete* G *- metric space* (X,G) *and* $\{T_n\}_{n\in N} : S \to S$ *be a sequence of mappings such that for all* $x,y,z \in X$ *and* $i,j,k \in N$,

$$\begin{aligned}F(&G(T_ix,T_jy,T_kz), G(x,y,z), G(T_ix,y,z),\\ &G(x,T_jy,z), G(x,y,T_kz)) \leq 0\end{aligned} \tag{1}$$

where $F \in \mathcal{F}_Q$. *Then* $\{T_n\}$ *has a unique common fixed point.*

Proof. Let $x_0 \in S$ be any arbitrary point. Define a sequence $\{x_n\}$ in S such that $x_{n+1} = T_{n+1}x_n$, $n = 0,1,2,...$. By (1) we have successively:

$$\begin{aligned}F(&G(T_nx_{n-1}, T_{n+1}x_n, T_{n+2}x_{n+1}), G(x_{n-1},x_n,x_{n+1}), G(T_nx_{n-1},x_n,x_{n+1}),\\ &G(x_{n-1},T_{n+1}x_n,x_{n+1}), G(x_{n-1},x_n,T_{n+2}x_{n+1})) \leq 0,\end{aligned}$$

$$F(G(x_n, x_{n+1}, x_{n+2}), G(x_{n-1}, x_n, x_{n+1}), G(x_n, x_n, x_{n+1}), \\ G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_{n+2})) \leq 0. \tag{2}$$

By (G_3) we have

$$\begin{cases} G(x_n, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}) \text{ and} \\ G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_{n+1}). \end{cases} \tag{3}$$

By (G_4) and (G_3) we have

$$G(x_{n-1}, x_n, x_{n+2}) \leq G(x_{n-1}, x_n, x_n) + G(x_n, x_n, x_{n+2}) \\ \leq G(x_{n-1}, x_n, x_{n+1}) + G(x_n, x_{n+1}, x_{n+2}). \tag{4}$$

Then by (3), (4) and (F_1) we have

$$F(G(x_n, x_{n+1}, x_{n+2}), G(x_{n-1}, x_n, x_{n+1}), G(x_n, x_{n+1}, x_{n+2}), \\ G(x_{n-1}, x_n, x_{n+1}), G(x_{n-1}, x_n, x_{n+1}) + G(x_n, x_{n+1}, x_{n+2})) \leq 0. \tag{5}$$

By (F_2) and (5) we obtain

$$G(x_n, x_{n+1}, x_{n+2}) \leq hG\left(x_{n-1}, x_n, x_{n+1}\right)$$

which implies

$$G(x_n, x_{n+1}, x_{n+2}) \leq h^n G\left(x_0, x_1, x_2\right).$$

Now for any positive integers $k \geq m \geq n \geq 1$ we obtain

$$\begin{aligned} G\left(x_n, x_m, x_k\right) &\leq G\left(x_n, x_{n+1}, x_{n+2}\right) + G\left(x_{n+1}, x_{n+2}, x_{n+3}\right) + ... + G\left(x_{k-2}, x_{k-1}, x_k\right) \\ &\leq h^n \left(1 + h + ... + h^{k-2}\right) G\left(x_0, x_1, x_2\right) \\ &\leq \frac{h^n}{1-h} G\left(x_0, x_1, x_2\right) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is a G - Cauchy sequence. Since (X, G) is a complete G - metric space, there exists a point $u \in X$ such that $\{x_n\}$ converges to u. Since $\{x_n\} \subset S$ and S is G - closed, then $u \in S$.

Now, we prove that u is a common fixed point for $\{T_i\}_{n \in N}$.

By (1) we have successively

$$F(G(T_n x_{n-1}, T_j u, T_k u), G(x_{n-1}, u, u), G(T_n x_{n-1}, u, u), \\ G(x_{n-1}, T_j u, u), G(x_{n-1}, u, T_k u)) \leq 0,$$

$$F(G(x_n, T_j u, T_k u), G(x_{n-1}, u, u), G(x_n, u, u), \\ G(x_{n-1}, T_j u, u), G(x_{n-1}, u, T_k u)) \leq 0.$$

Letting n tends to infinity we obtain

$$F(G(u, T_j u, T_k u), 0, 0, G(u, T_j u, u), G(u, u, T_k u)) \leq 0.$$

By (G_3) we have

$$G\left(u, T_j u, u\right) \leq G\left(u, T_j u, T_k u\right)$$

and

$$G\left(u, u, T_k u\right) \leq G\left(u, T_j u, T_k u\right).$$

By (F_1) we obtain

$$F(G(u, T_j u, T_k u), 0, 0, G(u, T_j u, T_k u), G(u, T_j u, T_k u)) \leq 0.$$

By (F_3) we obtain $G(u, T_j u, T_k u) = 0$, i.e. $u = T_j u = T_k u$. Hence, u is a common fixed point for $\{T_n\}_{n\in N}$.

Suppose that v is another common fixed point of $\{T_n\}_{n\in N}$.

By (1) we have successively

$$F(G(T_i u, T_j u, T_k v), G(u, u, v), G(T_i u, u, v), G(u, T_j u, v), G(u, u, T_k v)) \leq 0,$$

$$F(G(u, u, v), G(u, u, v), G(u, u, v), G(u, u, v), G(u, u, v)) \leq 0.$$

By (F_1) we have

$$F(G(u, u, v), G(u, u, v), G(u, u, v), G(u, u, v), G(u, u, v) + G(u, u, v)) \leq 0,$$

which implies

$$G(u, u, v) \leq hG(u, u, v),$$

hence

$$G(u, u, v) = 0,$$

i.e. $u = v$ and u is the unique common fixed point of $\{T_n\}_{n\in N}$.

Theorem 4.2. *Let S be a closed subset of a complete G - metric space (X, G) and $\{T_n\}_{n\in N} : S \to S$ be a sequence of mappings such that for all $x, y, z \in X$ and $i, j, k \in N$,*

$$F(G(T_i x, T_j y, T_k z), G(x, y, z), G(T_i x, y, T_k z), G(T_i x, z, T_j y), G(T_j y, T_k z, x)) \leq 0 \tag{6}$$

where $F \in \mathcal{F}_Q$. Then $\{T_n\}_{n\in N}$ has an unique common fixed point.

Proof. The proof is similar to the proof of Theorem 4.1.

Remark 4.3. *By Theorem 4.1, 4.2 and Examples 3.2 - 3.9 we obtain new particular results.*

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