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Global boundedness for vector valued minimizers of some anisotropic variational integrals

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Abstract

This paper deals with anisotropic integral functionals of the type

$$\mathcal{I}(u;\Omega) = \int_\Omega f(x, Du(x))dx.$$

We present a monotonicity inequality on the density $f(x,\xi)$ with weight, which guarantees global boundedness of minimizers u with gradient constraints.

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1 Introduction and Statement of Results.

Let $\Omega \subset \mathrm{R}^n (n \geq 2)$ be a bounded domain. For $p_1, \cdots, p_n \in (1, +\infty)$, we let $\bar{p} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}\right)^{-1}$ and $p_i' = \frac{p_i}{p_i - 1}$ be the harmonic mean of $p_1, \cdots, p_n$ and the Hölder conjugate of p_i, respectively.

For every $i \in \{1, \cdots, n\}$, we let ν_i to be a function on Ω such that $\nu_i > 0$ a.e. in Ω and

$$\nu_i \in L^1_{loc}(\Omega), \quad \frac{1}{\nu_i} \in L^{1/(p_i-1)}(\Omega). \tag{1.1}$$

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Denote by $W^{1,(p_i)}(\nu,\Omega)$ the set of all functions $u\in L^1(\Omega)$ such that $\nu_i|D_iu|^{p_i}\in L^1(\Omega)$. The norm for $u\in W^{1,(p_i)}(\nu,\Omega)$ is defined by

$$\|u\|_{1,(p_i),\nu}=\int_\Omega |u|dx+\sum_{i=1}^n\left(\int_\Omega \nu_i|D_iu|^{p_i}dx\right)^{1/p_i}.$$

It is known, by the second inclusion of (1.1), that the set $W^{1,(p_i)}(\nu,\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{1,(p_i),\nu}$. Moreover, by virtue of the first inclusion of (1.1), we have $C_0^\infty(\Omega)\subset W^{1,(p_i)}(\nu,\Omega)$. We denote by $W_0^{1,(p_i)}(\nu,\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in the norm of $W^{1,(p_i)}(\nu,\Omega)$. The set $W_0^{1,(p_i)}(\nu,\Omega)$ is a reflexive Banach space with respect to the norm induced by $\|\cdot\|_{1,(p_i),\nu}$. We denote by $W^{1,(p_i)}(w,\Omega,\mathrm{R}^N)$ the set of all vector valued functions $u=(u^1,\cdots,u^N)$ such that for every $j\in\{1,\cdots,N\}$ we have $u^j\in W^{1,(p_i)}(\nu,\Omega)$. In particular, $W^{1,(p_i)}(\Omega)$, $W_0^{1,(p_i)}(\Omega)$, $W^{1,(p_i)}(\Omega,\mathrm{R}^N)$ and $W_0^{1,(p_i)}(\Omega,\mathrm{R}^N)$ stand for the special cases of $W^{1,(p_i)}(\nu,\Omega)$, $W_0^{1,(p_i)}(\nu,\Omega)$, $W^{1,(p_i)}(\nu,\Omega,\mathrm{R}^N)$ and $W_0^{1,(p_i)}(\nu,\Omega,\mathrm{R}^N)$ with $\nu_i\equiv 1$, $i=1,\cdots,n$, respectively.

For a vector $m=(m_1,\cdots,m_n)\in\mathrm{R}^n$ with $m_i>0$, $i=1,\cdots,n$, we set

$$q_m=n\left(\sum_{i=1}^n\frac{1+m_i}{m_ip_i}-1\right)^{-1}.$$

We consider the anisotropic integral functional

$$\mathcal{I}(u;\Omega)=\int_\Omega f(x,Du(x))dx, \tag{1.2}$$

where $f:\Omega\times\mathrm{R}^{N\times n}\to\mathrm{R}$ is a Carathéodory function. We assume that there exist a constant $\mu>0$ and a function $M(x)\in L^r(\Omega)$, $r\geq 1$, such that

$$f(x,\tilde A)+\mu\sum_{i=1}^n\nu_i|\tilde A_i-A_i|^{p_i}\leq f(x,A)+M(x) \tag{1.3}$$

for every pair of matrices $\tilde A, A\in\mathrm{R}^{N\times n}$ such that there exists a row β with $\tilde A^\beta=0$ and for every remaining row $\alpha\neq\beta$ we have $\tilde A^\alpha=A^\alpha$.

We let $\varphi:\Omega\to\mathrm{R}$ be a nonnegative function and $u_*\in W^{1,(p_i)}(w,\Omega,\mathrm{R}^N)$ be such that $|Du_*(x)|\leq\varphi(x)$, a.e. Ω. We assume that for every $x\in\Omega$,

$$K(x)=\{\xi\in\mathrm{R}^{N\times n}:|\xi|\leq\varphi(x)\}.$$

We define

$$V(u_*,K)=\{v\in u_*+W_0^{1,(p_i)}(w,\Omega,\mathrm{R}^N):Dv(x)\in K(x)\text{ for a.e. }x\in\Omega\}.$$

It is obvious that $Du_*(x)\in K(x)$ for a.e. $x\in\Omega$. Therefore, $V(u_*,K)\neq\emptyset$. It is easy to see that the set $V(u_*,K)$ is convex and closed in $W^{1,(p_i)}(w,\Omega,\mathrm{R}^N)$.

The main result of this paper is the following theorem.

Theorem 1.1 *Let $m \in R^n$, and let the following two conditions be satisfied:*
(a) for every $i \in \{1, \cdots, n\}$ we have $m_i \geq 1/(p_i - 1)$ and $1/w_i \in L^{m_i}(\Omega)$;
(b) $q_m > \bar{p}$.
We consider the integral functional (1.2) under the monotonicity inequality (1.3). We let $u \in W^{1,(p_i)}(w, \Omega, R^N)$ be such that

$$u \in V(u_*, K), \tag{1.4}$$

$$\forall v \in V(u_*, K), \ \int_\Omega f(x, Du(x))dx \leq \int_\Omega f(x, Dv(x))dx. \tag{1.5}$$

Then, for every component u^β of u, we have

$$\inf_{\partial\Omega} u_*^\beta(x) - c_* \leq u^\beta(x) \leq \sup_{\partial\Omega} u_*^\beta(x) + c_* \tag{1.6}$$

for almost every $x \in \Omega$, where

$$c_* = c \left(\frac{\|M\|_{L^r(\Omega)}}{\mu} \right)^{\frac{1}{\bar{p}}} |\Omega|^{\left[(1-\frac{1}{r})\frac{q_m}{\bar{p}}-1\right]\frac{1}{q_m}} 2^{(1-\frac{1}{r})\frac{q_m}{\bar{p}}\left[(1-\frac{1}{r})\frac{q_m}{\bar{p}}-1\right]^{-1}},$$

where $|\Omega|$ is the n-dimensional Lebesgue measure of Ω, and c and μ are the constants from (2.1) and (1.3), respectively.

Remark 1.2 *We refer the readers to [1-6] for some related results.*

A model density f for the monotonicity inequality (1.3) is given in the following.

Theorem 1.3 *For every $i = 1, \cdots, n$, let us consider $p_i \geq 2$ and $a_i > 0$; we take $m(x) \geq 0$, a.e. $x \in \Omega$. Let us consider $f : \Omega \times R^{N\times n} \to R$ defined as follows:*

$$f(x, A) = \sum_{i=1}^n a_i \nu_i |A_i|^{p_i} + m(x) h \left(\frac{1}{1 + \|A\|} \right),$$

where

$$\|A\| = \left(Tr(A^t A) \right)^{1/2} = \left(\sum_{i=1}^n \sum_{j=1}^N |A_i^j|^2 \right)^{1/2}$$

is the Hilbert-Schmidt norm of the matrix $A = (A_i^j)$, and $h(x) : (0, +\infty) \to R$ is a Lipschitz continuous function:

$$|h(t_1) - h(t_2)| \leq C|t_1 - t_2|, \ \forall t_1, t_2 \geq 0. \tag{1.7}$$

Then the monotonicity inequality (1.3) holds true with $\mu = \min_{1\leq i\leq n}\{a_i\}$ and $M(x) = Cm(x)$, where C is the constant in (1.7).

2 Proof of Theorems 1.1 and 1.3.

In order to prove Theorem 1.1, we need two preliminary lemmas.

The following lemma is the Sobolev Imbedding Theorem with weight, which comes from [7, Proposition 2.1], the proof can be found in [8].

Lemma 2.1 *Let $m \in R^n$, and let the following conditions be satisfied: for every $i \in \{1, \cdots, n\}$ we have $m_i \geq 1/(p_i - 1)$ and $1/w_i \in L^{m_i}(\Omega)$. Then $W_0^{1,(p_i)}(w, \Omega) \subset L^{q_m}(\Omega)$, and there exists a positive constant c such that for every function $v \in W_0^{1,(p_i)}(w, \Omega)$,*

$$\left(\int_\Omega |v|^{q_m} dx\right)^{1/q_m} \leq c \prod_{i=1}^n \left(\int_\Omega w_i |D_i v|^{p_i} dx\right)^{1/np_i}. \tag{2.1}$$

The next lemma comes from [9, Lemma 4.1].

Lemma 2.2 *Let $\chi : [t_0, +\infty) \to [0, +\infty)$ be non-increasing. We assume that there exist $\tilde{C}, a > 0$ and $b > 1$ such that*

$$t_0 \leq t < T \Rightarrow \chi(T) \leq \frac{\tilde{C}}{(T-t)^a} [\chi(t)]^b.$$

Then it results that

$$\chi(t_0 + d) = 0,$$

where

$$d = \left[\tilde{C} \left(\chi(t_0)\right)^{b-1} 2^{\frac{ab}{b-1}}\right]^{\frac{1}{a}}.$$

Proof of Theorem 1.1. As in the proof of Lemma 2.1 in [1], we define $I_{\beta,t} : \mathrm{R}^N \to \mathrm{R}^N$ as follows:

$$\forall y = (y^1, \cdots, y^N) \in \mathrm{R}^N, \quad I_{\beta,t}(y) = (I^1_{\beta,t}(y), I^2_{\beta,t}(y), \cdots, I^N_{\beta,t}(y))$$

with

$$I^\alpha_{\beta,t}(y) = \begin{cases} y^\alpha, & \alpha \neq \beta \\ y^\beta \wedge t = \min\{y^\beta, t\}, & \alpha = \beta. \end{cases}$$

For $u \in V(u_*, K)$, we need to show that $I_{\beta,t}(u) \in V(u_*, K)$. In fact, it is obvious that $I_{\beta,t}(u) \in u_* + W_0^{1,(p_i)}(w, \Omega, \mathrm{R}^N)$; in order to show that $DI_{\beta,t}(u) \in K(x)$, it is sufficient to derive $|DI_{\beta,t}(u)(x)| \leq K(x)$. This is true because

$$D_i I^\alpha_{\beta,t}(u) = \begin{cases} D_i u^\alpha, & \alpha \neq \beta, \\ D_i u^\beta 1_{\{u^\beta \leq t\}}, & \alpha = \beta, \end{cases} \tag{2.1}$$

where 1_B is the characteristic function of the set B, that is, $1_B(x) = 1$ if $x \in B$ and $1_B(x) = 0$ otherwise.

Our next goal is to show that, for every $u = (u^1, u^2, \cdots, u^N) \in W^{1,(p_i)}(w, \Omega, \mathrm{R}^N)$, for any $\beta \in \{1, 2, \cdots, N\}$, for all $t \in \mathrm{R}$, the following inequality holds true

$$\mathcal{I}(I_{\beta,t}(u)) + \mu \sum_{i=1}^{n} \int_\Omega w_i |D_i(I_{\beta,t}(u)) - D_i u|^{p_i} dx \le \mathcal{I}(u) + \int_{\{u^\beta > t\}} M(x)dx. \quad (2.2)$$

Indeed, on $\{x \in \Omega : u^\beta > t\}$ we have $D(I^\beta_{\beta,t}(u)) = 0$, and for $\alpha \neq \beta$, $D(I^\alpha_{\beta,t}(u)) = D_i u^\alpha$; so we can apply (1.3) with $\tilde{A} = D(I_{\beta,t}(u))$ and $A = Du$; we obtain

$$f(x, D(I_{\beta,t}(u))) + \mu \sum_{i=1}^{n} w_i |D_i(I_{\beta,t}(u)) - D_i u|^{p_i} \le f(x, Du) + M(x) \qquad (2.3)$$

for $x \in \{x \in \Omega : u^\beta > t\}$. On $\{x \in \Omega : u^\beta \le t\}$, $D(I_{\beta,t}(u)) = Du$, thus

$$f(x, D(I_{\beta,t}(u))) + \mu \sum_{i=1}^{n} w_i |D_i(I_{\beta,t}(u)) - D_i u|^{p_i} dx = f(x, Du) \qquad (2.4)$$

for $x \in \{x \in \Omega : u^\beta \le t\}$. From (2.3) and (2.4) we have

$$f(x, DI_{\beta,t}(u)) + \mu \sum_{i=1}^{n} w_i |D_i(I_{\beta,t}(u)) - D_i u|^{p_i} \le f(x, Du) + M(x) \cdot 1_{\{u^\beta > t\}}. \quad (2.5)$$

Since $\mathcal{I}(u) < +\infty$, then $f(x, Du(x)) \in L^1(\Omega)$, thus $f(x, DI_{\beta,t}(u)) \in L^1(\Omega)$ too. Integrating (2.5) with respect to x, we get (2.2).

Let us fix $\beta \in \{1, 2, \cdots, N\}$. If $\sup_{\partial\Omega} u^\beta_*(x) = +\infty$ then the right hand side of (1.6) is satisfied. Thus we assume $\sup_{\partial\Omega} u^\beta_*(x) < t_0 < t < +\infty$ and we note that under this assumption $I_{\beta,t}(u) \in u + W^{1,1}_0(w, \Omega, \mathrm{R}^N)$ and $D_i(I_{\beta,t}(u)) \in L^{p_i}(w, \Omega, \mathrm{R}^N)$, $i \in \{1, \cdots, n\}$, this is because

$$u^\beta \wedge t = \min\{u^\beta, t\} = u^\beta - [\max\{u^\beta - t, 0\}] = u^\beta - [(u^\beta - t) \vee 0] = u^\beta - \phi,$$

where $\phi = \max\{u^\beta - t, 0\} = (u^\beta - t) \vee 0 \in W^{1,1}_0(\Omega)$ and $D_i\phi = D_i u^\beta \cdot 1_{\{u^\beta > t\}} \in L^{p_i}(w_i, \Omega)$, $i = 1, 2, \cdots, n$. From (1.5) and (2.2) it results that

$$\mathcal{I}(u) \le \mathcal{I}(I_{\beta,t}(u)) \le \mathcal{I}(u) - \mu \sum_{i=1}^{n} \int_\Omega w_i |D_i(I_{\beta,t}(u)) - D_i u|^{p_i} dx + \int_{\{u^\beta > t\}} M(x)dx,$$

from which we derive

$$\mu \sum_{i=1}^{n} \int_\Omega w_i |D_i\phi|^{p_i} dx = \mu \sum_{i=1}^{n} \int_\Omega w_i |D_i(I_{\beta,t}(u)) - D_i u|^{p_i} dx \le \int_{\{u^\beta > t\}} M(x)dx. \quad (2.6)$$

If $r < +\infty$, we apply Hölder inequality and we get

$$\int_{\{u^\beta > t\}} M(x)dx \le \|M\|_{L^r(\Omega)} |\{u^\beta > t\}|^{1-\frac{1}{r}}.$$

If $r = +\infty$, then

$$\int_{\{u^\beta > t\}} M(x)dx \le \|M\|_{L^\infty(\Omega)} |\{u^\beta > t\}| = \|M\|_{L^r(\Omega)} |\{u^\beta > t\}|^{1-\frac{1}{r}}.$$

In both cases, from (2.6) it results that

$$\sum_{i=1}^n \int_\Omega w_i |D_i\phi|^{p_i} dx \le \frac{\|M\|_{L^r(\Omega)}}{\mu} |\{u^\beta > t\}|^{1-\frac{1}{r}}.$$

We apply Lemma 2.1 and we get

$$\begin{aligned} & \left(\int_{\{u^\beta > t\}} |u^\beta - t|^{q_m} dx\right)^{1/q_m} \\ = & \left(\int_{\{u^\beta > t\}} |\phi|^{q_m} dx\right)^{1/q_m} = \left(\int_\Omega |\phi|^{q_m} dx\right)^{1/q_m} \\ \le & \ c \prod_{i=1}^n \left(\int_\Omega w_i |D_i\phi|^{p_i} dx\right)^{1/np_i} \le c\left(\frac{\|M\|_{L^r(\Omega)}}{\mu} |\{u^\beta > t\}|^{1-\frac{1}{r}}\right)^{\frac{1}{\bar{p}}}. \end{aligned} \tag{2.7}$$

For $T > t$ we have

$$\begin{aligned} (T-t)^{q_m} |\{u^\beta > T\}| &= \int_{\{u^\beta > T\}} (T-t)^{q_m} dx \\ \le \int_{\{u^\beta > T\}} (u^\beta - t)^{q_m} dx &\le \int_{\{u^\beta > t\}} (u^\beta - t)^{q_m} dx. \end{aligned} \tag{2.8}$$

From (2.7) and (2.8) we get

$$|\{u^\beta > T\}| \le c^{q_m} \left(\frac{\|M\|_{L^r(\Omega)}}{\mu}\right)^{\frac{q_m}{\bar{p}}} \frac{1}{(T-t)^{q_m}} |\{u^\beta > t\}|^{(1-\frac{1}{r})\frac{q_m}{\bar{p}}}$$

for every T, t with $T > t \ge t_0$. We set $\chi(t) = |\{u^\beta > t\}|$, $\tilde{C} = c^{q_m}\left(\frac{\|M\|_{L^r(\Omega)}}{\mu}\right)^{\frac{q_m}{\bar{p}}}$, $a = q_m$ and $b = (1-\frac{1}{r})\frac{q_m}{\bar{p}}$. We use Lemma 2.2 and we get $|\{u^\beta > t_0 + c_*\}| = 0$, that is, $u^\beta \le t_0 + c_*$ almost everywhere in Ω, where

$$c_* = c\left(\frac{\|M\|_{L^r(\Omega)}}{\mu}\right)^{\frac{1}{\bar{p}}} |\Omega|^{\left[(1-\frac{1}{r})\frac{q_m}{\bar{p}}-1\right]\frac{1}{q_m}} 2^{(1-\frac{1}{r})\frac{q_m}{\bar{p}}\left[(1-\frac{1}{r})\frac{q_m}{\bar{p}}-1\right]^{-1}}.$$

In order to get the right hand side of (1.6), we take a sequence $\{(t_0)_m\}_m$ with $(t_0)_m \to \sup_{\partial\Omega} u^\beta$. We apply the right hand side of (1.6) to $-u$ and we get the left hand side of (1.7). This ends the proof of Theorem 1.1.

Proof of Theorem 1.3. We assume that $\tilde{A}, A \in \mathrm{R}^{N\times n}$ with $\tilde{A}^\beta = 0$ and $\tilde{A}^\alpha = A^\alpha$ for $\alpha \neq \beta$. Then

$$\sum_\alpha |A_i^\alpha|^2 = \sum_\alpha |A_i^\alpha - \tilde{A}_i^\alpha|^2 + \sum_\alpha |\tilde{A}_i^\alpha|^2.$$

Thus

$$|A_i|^2 = |A_i - \tilde{A}_i|^2 + |\tilde{A}_i|^2.$$

The conditions $p_i \geq 2$, $i = 1, \cdots, n$, imply

$$|A_i|^{p_i} \geq |A_i - \tilde{A}_i|^{p_i} + |\tilde{A}_i|^{p_i}.$$

Thus

$$\begin{aligned}
& f(x,\tilde{A}) + \min_{1\leq i\leq n}\{b_i\} \cdot \sum_{i=1}^n w_i |\tilde{A}_i - A_i|^{p_i} \\
\leq\ & \sum_{i=1}^n b_i w_i |\tilde{A}_i|^{p_i} + m(x) h\left(\frac{1}{1+\|\tilde{A}\|}\right) + \sum_{i=1}^n b_i w_i |\tilde{A}_i - A_i|^{p_i} \\
\leq\ & \sum_{i=1}^n b_i w_i |A_i|^{p_i} + m(x) h\left(\frac{1}{1+\|\tilde{A}\|}\right) \\
=\ & \sum_{i=1}^n b_i w_i |A_i|^{p_i} + m(x) h\left(\frac{1}{1+\|A\|}\right) + m(x)\left[h\left(\frac{1}{1+\|\tilde{A}\|}\right) - h\left(\frac{1}{1+\|A\|}\right)\right] \\
\leq\ & f(x,A) + Cm(x)\left|\frac{1}{1+\|\tilde{A}\|} - \frac{1}{1+\|A\|}\right| \\
=\ & f(x,A) + Cm(x)\left(\frac{|A^\beta|}{(1+\|A\|)(1+\|\tilde{A}\|)}\right) \\
\leq\ & f(x,A) + Cm(x).
\end{aligned}$$

Thus the monotonicity inequality (1.5) holds true with $\mu = \min_{1\leq i\leq n}\{a_i\}$ and $M(x) = Cm(x)$.

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