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Derivation Rings of Centerless Perfect Lie Rings Are Complete

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Abstract

It is proved that the derivation ring of a centerless perfect Lie ring of arbitrary dimension is complete and that the holomorph of a centerless perfect Lie ring is complete if and only if its outer derivation ring is centerless.

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1 Introduction

A Lie ring is called *complete* if its center is zero, and all its derivations are inner. In [5], Liao Jun proved a theorem, the so-called *derivation tower theorem*, that the last term of the derivation tower of a centerless Lie ring is complete. It is also known that the holomorph of a Lie ring is complete. Suggested by the derivation tower theorem, complete Lie rings occur naturally in the study of Lie rings and would provide interesting objects for investigation. Robinson [8] presented some interesting examples of complete Lie rings.

It is known that a simple Lie ring is not always complete. In this case, a natural question is: *How many is the length of the derivation tower of a Lie ring?*

In this paper, we answer the above question. Precisely, we prove that the derivation ring of a *centerless perfect* Lie ring (i.e., a Lie ring $\mathfrak{g}$ with zero center and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$) of arbitrary dimension is complete, thus as a consequence, the length of the derivation tower of any simple Lie ring is ≤ 1.

2 Preliminary Notes

First we recall the definitions of a derivation and the holomorph of a Lie ring $\mathfrak{g}$.

Definition 2.1 *[5] A derivation of a Lie ring* $\mathfrak{g}$ *is an endomorphism* $d : \mathfrak{g} \to \mathfrak{g}$ *of the additive group of* $\mathfrak{g}$ *such that*

$$d([x, y]) = [d(x), y] + [x, d(y)] \quad \text{for} \quad x, y \in \mathfrak{g}. \tag{2.1}$$

We denote by $\operatorname{Der}\mathfrak{g}$ *the vector space of derivations of* $\mathfrak{g}$*, which forms a Lie ring with respect to the commutator of endomorphism, called the derivation ring of* $\mathfrak{g}$*.*

Clearly, the space $\operatorname{ad}_{\mathfrak{g}} = \{\operatorname{ad}_x \mid x \in \mathfrak{g}\}$ of *inner derivations* is an ideal of $\operatorname{Der}\mathfrak{g}$. We call $\operatorname{Der}\mathfrak{g}/\operatorname{ad}_{\mathfrak{g}}$ the *outer derivation ring* of $\mathfrak{g}$.

Definition 2.2 *[5] The holomorph* $\mathfrak{h}(\mathfrak{g})$ *of a Lie ring* $\mathfrak{g}$ *is the direct sum of the vector spaces* $\mathfrak{h}(\mathfrak{g}) = \mathfrak{g} \oplus \operatorname{Der}\mathfrak{g}$ *with the following operations*

$$(x, d) + (y, e) = (x + y, d + e), \tag{2.2}$$

$$[(x, d), (y, e)] = ([x, y] + d(y) - e(x), [d, e]) \quad \text{for} \quad x, y \in \mathfrak{g},\ d, e \in \operatorname{Der}\mathfrak{g}. \tag{2.3}$$

An element (x, d) *of* $\mathfrak{h}(\mathfrak{g})$ *is also written as* $x + d$*.*

Obviously, $\mathfrak{g}$ is an ideal of $\mathfrak{h}(\mathfrak{g})$ and $\mathfrak{h}(\mathfrak{g})/\mathfrak{g} \cong \operatorname{Der}\mathfrak{g}$. Thus we write

$$\mathfrak{h}(\mathfrak{g}) = \mathfrak{g} \rtimes \operatorname{Der}\mathfrak{g}. \tag{2.4}$$

For a Lie ring $\mathfrak{g}$, we denote by $C(\mathfrak{g})$ the center of $\mathfrak{g}$, i.e., $C(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] = 0\}$.

3 Main Results

The main result of this paper is the following.

Theorem 3.1 *Let* $\mathfrak{g}$ *be a perfect Lie ring with zero center. Then we have*

(i) The derivation ring $\operatorname{Der}\mathfrak{g}$ *is complete.*

(ii) The holomorph $\mathfrak{h}(\mathfrak{g})$ *is complete if and only if the center of outer derivation ring is zero, i.e.,* $C(\operatorname{Der}\mathfrak{g}/ad_{\mathfrak{g}}) = 0$*.*

Proof. (i) Assume that $d \in C(\operatorname{Der}\mathfrak{g})$. Then in particular we have $[d, \operatorname{ad}_x](y) = 0$ for all $x, y \in \mathfrak{g}$. Thus $d([x, y]) = [x, d(y)]$. Hence by (2.1), $[d(x), y] = 0$ for all $x, y \in \mathfrak{g}$. Since $\mathfrak{g}$ has zero center, we obtain $d(x) = 0$, i.e., $d = 0$. Therefore

$$C(\operatorname{Der}\mathfrak{g}) = 0. \tag{3.1}$$

Now we prove that all derivations of the Lie ring $\mathrm{Der}\,\mathfrak{g}$ are inner. First we have

Claim 1. Let $D \in \mathrm{Der}\,(\mathrm{Der}\,\mathfrak{g})$. If $D(\mathrm{ad}_{\mathfrak{g}}) = 0$, then $D = 0$.

Let $d \in \mathrm{Der}\,\mathfrak{g}$, $x \in \mathfrak{g}$. Note from (2.1) that in the Lie ring $\mathrm{Der}\,\mathfrak{g}$, we have

$$[d, \mathrm{ad}_x] = \mathrm{ad}_{d(x)} \in \mathrm{ad}_{\mathfrak{g}}. \tag{3.2}$$

Using this, noting that $D(d) \in \mathrm{Der}\,\mathfrak{g}$ and the fact that $D(\mathrm{ad}_{\mathfrak{g}}) = 0$, we have

$$\begin{aligned} \mathrm{ad}_{D(d)(x)} &= [D(d), \mathrm{ad}_x] \\ &= D([d, \mathrm{ad}_x]) - [d, D(\mathrm{ad}_x)] \\ &= D(\mathrm{ad}_{d(x)}) \\ &= 0. \end{aligned} \tag{3.3}$$

Since $\mathfrak{g}$ has zero center, (3.3) gives $D(d)(x) = 0$ for all $x \in \mathfrak{g}$, which means that $D(d) = 0$ as a derivation of $\mathfrak{g}$. But d is arbitrary, we obtain $D = 0$. This proves the claim.

Since $\mathfrak{g}$ is perfect, for any $x \in \mathfrak{g}$, we can write x as

$$x = \sum_{i \in I} [x_i, y_i] \quad \text{for some} \quad x_i, y_i \in \mathfrak{g}, \tag{3.4}$$

and for some finite index set I. Then

$$\mathrm{ad}_x = \sum_{i \in I} [\mathrm{ad}_{x_i}, \mathrm{ad}_{y_i}]. \tag{3.5}$$

Then for any $D \in \mathrm{Der}(\mathrm{Der}\,\mathfrak{g})$, we have

$$\begin{aligned} D(\mathrm{ad}_x) &= \sum_{i \in I} D([\mathrm{ad}_{x_i}, \mathrm{ad}_{y_i}]) \\ &= \sum_{i \in I} ([D(\mathrm{ad}_{x_i}), \mathrm{ad}_{y_i}] + [\mathrm{ad}_{x_i}, D(\mathrm{ad}_{y_i})]). \end{aligned} \tag{3.6}$$

Let $d_i = D(\mathrm{ad}_{x_i})$, $e_i = D(\mathrm{ad}_{y_i}) \in \mathrm{Der}\,\mathfrak{g}$. Then by (3.2), we have

$$\begin{aligned} D(\mathrm{ad}_x) &= \sum_{i \in I} (\mathrm{ad}_{d_i(y_i)} - \mathrm{ad}_{e_i(x_i)}) \\ &= \mathrm{ad}_{\sum_{i \in I}(d_i(y_i) - e_i(x_i))} \in \mathrm{ad}_{\mathfrak{g}}. \end{aligned} \tag{3.7}$$

This means that $D(\mathrm{ad}_x) = \mathrm{ad}_y$ for some $y \in \mathfrak{g}$. Since $C(\mathfrak{g}) = 0$, such y is unique. Thus $d : x \mapsto y$ defines an endomorphism of $\mathfrak{g}$ such that

$$D(\mathrm{ad}_x) = \mathrm{ad}_{d(x)}. \tag{3.8}$$

For $x, y \in \mathfrak{g}$, we have

$$\begin{aligned}\operatorname{ad}_{d(x+y)} &= D(\operatorname{ad}_{x+y}) \\ &= D(\operatorname{ad}_x + \operatorname{ad}_y) \\ &= D(\operatorname{ad}_x) + D(\operatorname{ad}_y) \\ &= \operatorname{ad}_{d(x)} + \operatorname{ad}_{d(y)} \\ &= \operatorname{ad}_{d(x)+d(y)},\end{aligned} \tag{3.9}$$

$$\begin{aligned}\operatorname{ad}_{d([x,y])} &= D(\operatorname{ad}_{[x,y]}) \\ &= D([\operatorname{ad}_x, \operatorname{ad}_y]) \\ &= [D(\operatorname{ad}_x), \operatorname{ad}_y] + [\operatorname{ad}_x, D(\operatorname{ad}_y)] \\ &= [\operatorname{ad}_{d(x)}, \operatorname{ad}_y] + [\operatorname{ad}_x, \operatorname{ad}_{d(y)}] \\ &= \operatorname{ad}_{[d(x),y]+[x,d(y)]}.\end{aligned} \tag{3.10}$$

This and the fact that $C(\mathfrak{g}) = 0$ mean that $d(x + y) = d(x) + d(y)$ and $d([x, y]) = [d(x), y] + [x, d(y)]$, i.e., $d \in \operatorname{Der} \mathfrak{g}$. Then (2.8) gives that $D(\operatorname{ad}_x) = \operatorname{ad}_{d(x)} = [d, \operatorname{ad}_x]$ for all $x \in \mathfrak{g}$, i.e.,

$$(D - \operatorname{ad}_d)(\operatorname{ad}_{\mathfrak{g}}) = 0. \tag{3.11}$$

By Claim I, we have $D - \operatorname{ad}_d = 0$, i.e., $D = \operatorname{ad}_d$ is an inner derivation on $\operatorname{Der} \mathfrak{g}$. This together with (3.1) proves Theorem 3.1(i).

(ii) "$\Longleftarrow$": First we prove the sufficiency. So suppose $C(\operatorname{Der} \mathfrak{g}/\operatorname{ad}_{\mathfrak{g}}) = 0$. We want to prove $\mathfrak{h}(\mathfrak{g})$ is complete.

First we prove $C(\mathfrak{h}(\mathfrak{g})) = 0$. Suppose $h = x + d \in C(\mathfrak{h}(\mathfrak{g}))$ for some $x \in \mathfrak{g}$, $d \in \operatorname{Der} \mathfrak{g}$. Letting $y = 0$ in (2.3), we obtain $[d, e] = 0$ for all $e \in \operatorname{Der} \mathfrak{g}$, i.e., $d \in C(\operatorname{Der} \mathfrak{g}) = 0$. Then $h = x \in C(\mathfrak{h}(\mathfrak{g})) \cap \mathfrak{g} \subset C(\mathfrak{g}) = 0$, i.e., $h = 0$. Thus $C(\mathfrak{h}(\mathfrak{g})) = 0$. Then as in the proof of (3.1), we have

$$C(\operatorname{Der}(\mathfrak{h}(\mathfrak{g}))) = 0. \tag{3.12}$$

Now let $\mathcal{D} \in \operatorname{Der}(\mathfrak{h}(\mathfrak{g}))$. For any $x \in \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$ written in the form (3.4), since $\mathfrak{g}$ is an ideal of $\mathfrak{h}(\mathfrak{g})$, we have

$$\mathcal{D}(x) = \sum_{i \in I} ([\mathcal{D}(x_i), y_i] + [x_i, \mathcal{D}(y_i)]) \in \mathfrak{g}. \tag{3.13}$$

Thus $d' = \mathcal{D}|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ is a derivation of $\mathfrak{g}$, i.e., $d' \in \operatorname{Der} \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$. Let $\mathcal{D}_1 = \mathcal{D} - \operatorname{ad}_{d'} \in \operatorname{Der}(\mathfrak{h}(\mathfrak{g}))$. Then $\mathcal{D}_1(x) = \mathcal{D}(x) - [d', x] = \mathcal{D}(x) - d'(x) = 0$, i.e.,

$$\mathcal{D}_1|_{\mathfrak{g}} = 0. \tag{3.14}$$

For any $d \in \operatorname{Der} \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$, by (2.4), we can write $\mathcal{D}_1(d) \in \mathfrak{h}(\mathfrak{g})$ as

$$\mathcal{D}_1(d) = x_d + d_1 \quad \text{for some} \quad x_d \in \mathfrak{g},\ d_1 \in \operatorname{Der} \mathfrak{g}. \tag{3.15}$$

Then in $\mathfrak{h}(\mathfrak{g})$, by (2.3), for any $y \in \mathfrak{g}$, we have

$$\begin{aligned} [x_d, y] + d_1(y) &= [x_d + d_1, y] \\ &= [\mathcal{D}_1(d), x_d] \\ &= \mathcal{D}_1([d, x_d]) - [d, \mathcal{D}_1(x_d)] \\ &= \mathcal{D}_1(d(x_d)) - [d, \mathcal{D}_1(x_d)] \\ &= 0, \end{aligned} \tag{3.16}$$

where the last equality follows from (3.14). This means that $d_1 = -\mathrm{ad}_{x_d}$ and so $\mathcal{D}_1(d) = x_d - \mathrm{ad}_{x_d}$. For $d, e \in \mathrm{Der}\,\mathfrak{g}$, we have

$$\begin{aligned} x_{d+e} - \mathrm{ad}_{x_{d+e}} &= \mathcal{D}_1((d+e)) \\ &= \mathcal{D}_1(d) + \mathcal{D}_1(e) \\ &= x_d - \mathrm{ad}_{x_d} + x_e - \mathrm{ad}_{x_e}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} x_{[d,e]} - \mathrm{ad}_{x_{[d,e]}} &= \mathcal{D}_1([d,e]) \\ &= [\mathcal{D}_1(d), e] + [d, \mathcal{D}_1(e)] \\ &= (-e(x_d) + d(x_e) + ([-\mathrm{ad}_{x_d}, e] + [d, -\mathrm{ad}_{x_e}]), \end{aligned} \tag{3.18}$$

where the last equality follows from (2.3). So

$$\mathrm{ad}_{x_{d+e}} = \mathrm{ad}_{x_d} + \mathrm{ad}_{x_e}, \tag{3.19}$$

$$\mathrm{ad}_{x_{[d,e]}} = [\mathrm{ad}_{x_d}, e] + [d, \mathrm{ad}_{x_e}], \tag{3.20}$$

i.e., the endomorphism $D : \mathrm{Der}\,\mathfrak{g} \to \mathrm{Der}\,\mathfrak{g}$ define by

$$D(d) = \mathrm{ad}_{x_d} \quad \text{for} \quad d \in \mathrm{Der}\,\mathfrak{g}, \tag{3.21}$$

is a derivation of $\mathrm{Der}\,\mathfrak{g}$. Since $\mathrm{Der}\,\mathfrak{g}$ is complete, there exists $d'' \in \mathrm{Der}\,\mathfrak{g}$ such that

$$D = \mathrm{ad}_{d''}. \tag{3.22}$$

For any $d \in \mathrm{Der}\,\mathfrak{g}$, we have

$$[d'', d] = D(d) = \mathrm{ad}_{x_d} \in \mathrm{ad}_{\mathfrak{g}}, \tag{3.23}$$

i.e., $d'' + \mathrm{ad}_{\mathfrak{g}} \in C(\mathrm{Der}\,\mathfrak{g}/\mathrm{ad}_{\mathfrak{g}}) = 0$. Thus $d'' \in \mathrm{ad}_{\mathfrak{g}}$. Therefore there exists $y \in \mathfrak{g}$ such that

$$d'' = \mathrm{ad}_y. \tag{3.24}$$

Note that $y - \mathrm{ad}_y \in \mathfrak{h}(\mathfrak{g})$. Let $\mathcal{D}_2 = \mathcal{D}_1 - \mathrm{ad}_{y - \mathrm{ad}_y}$. Then for any $x \in \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$, we have

$$\begin{aligned} \mathcal{D}_2(x) &= \mathcal{D}_1(x) - [y - \mathrm{ad}_y, x] \\ &= -[y, x] + \mathrm{ad}_y(x) \\ &= -[y, x] + [y, x] \\ &= 0, \end{aligned} \tag{3.25}$$

where the second equality follows from (3.14) and (2.3), and for any $d \in \operatorname{Der}\mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$,

$$\begin{aligned}\mathcal{D}_2(d) &= \mathcal{D}_1(d) - [y - \operatorname{ad}_y, d]\\ &= (x_d - \operatorname{ad}_{x_d}) - (-d(y) - [\operatorname{ad}_y, d])\\ &= (x_d + d(y)) - (\operatorname{ad}_{x_d} - [\operatorname{ad}_y, d])\\ &= 0,\end{aligned} \tag{3.26}$$

because $-\operatorname{ad}_{d(y)} = [\operatorname{ad}_y, d] = [d'', d] = \operatorname{ad}_{x_d}$ by (3.23) and (3.24), and $x_d = -d(y)$ (since $\mathfrak{g}$ has zero center). Thus (3.25) and (3.26) show that $\mathcal{D}_2 = 0$ and so $\mathcal{D}$ is inner. This and (3.12) prove that $\mathfrak{h}(\mathfrak{g})$ is complete.

" $\Longrightarrow$": Now we prove the necessity. So assume that $\mathfrak{h}(\mathfrak{g})$ is complete. Suppose conversely $C(\operatorname{Der}\mathfrak{g}/\operatorname{ad}_{\mathfrak{g}}) \neq 0$. Then there exists $D \in \operatorname{Der}\mathfrak{g}$ such that

$$D \notin \operatorname{ad}_{\mathfrak{g}} \quad \text{but} \quad [D, \operatorname{Der}\mathfrak{g}] \subset \operatorname{ad}_{\mathfrak{g}}. \tag{3.27}$$

Then for any $d \in \operatorname{Der}\mathfrak{g}$, there exists $x_d \in \mathfrak{g}$ such that $[D, d] = \operatorname{ad}_{x_d}$. Such x_d is unique since $C(\mathfrak{g}) = 0$. Using the facts that $[D, d+e] = [D, d] + [D, e]$ and $[D, [d, e]] = [[D, d], e] + [d, [D, e]]$, we obtain

$$x_{d+e} = x_d + x_e, \tag{3.28}$$

$$x_{[d,e]} = -e(x_d) + d(x_e) \quad \text{for} \quad d, e \in \operatorname{Der}\mathfrak{g}. \tag{3.29}$$

We define an endomorphism $\mathcal{D} : \mathfrak{h}(\mathfrak{g}) \to \mathfrak{h}(\mathfrak{g})$ as follows: $\mathcal{D}|_{\mathfrak{g}} = 0$, and for $d \in \operatorname{Der}\mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$, we define

$$\mathcal{D}(d) = x_d - \operatorname{ad}_{x_d} \in (\mathfrak{g} \rtimes \operatorname{Der}\mathfrak{g}) = \mathfrak{h}(\mathfrak{g}). \tag{3.30}$$

From this definition, we have

$$[\mathcal{D}(d), x] = 0 \quad \text{for} \quad x \in \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g}), d \in \operatorname{Der}\mathfrak{g} \subset \mathfrak{h}(\mathfrak{g}). \tag{3.31}$$

Then for any $h = x + d, h' = y + e \in \mathfrak{h}(\mathfrak{g})$, by (2.3) and the fact that $\mathcal{D}_{\mathfrak{g}} = 0$, we have

$$\begin{aligned}\mathcal{D}(h + h') &= \mathcal{D}(d + e)\\ &= x_{d+e} - \operatorname{ad}_{x_{d+e}}\\ &= x_d + x_e - \operatorname{ad}_{x_d} - \operatorname{ad}_{x_e}\\ &= x_d - \operatorname{ad}_{x_d} + x_e - \operatorname{ad}_{x_e}\\ &= \mathcal{D}(d) + \mathcal{D}(e)\\ &= \mathcal{D}(h) + \mathcal{D}(h'),\end{aligned} \tag{3.32}$$

$$\begin{aligned}\mathcal{D}([h,h']) &= \mathcal{D}([d,e]) \\ &= x_{[d,e]} - \mathrm{ad}_{x_{[d,e]}} \\ &= (-e(x_d) + d(x_e)) - ([\mathrm{ad}_{x_d}, e] + [d, \mathrm{ad}_{x_e}]) \\ &= (-e(x_d) - [\mathrm{ad}_{x_d}, e]) + (d(x_e) + [d, \mathrm{ad}_{x_e}]) \\ &= [\mathcal{D}(d), e] + [d, \mathcal{D}(e)] \\ &= [\mathcal{D}(h), h'] + [h, \mathcal{D}(h')].\end{aligned} \tag{3.33}$$

Thus $\mathcal{D}$ is a derivation of $\mathfrak{h}(\mathfrak{g})$. Since $\mathfrak{h}(\mathfrak{g})$ is complete, $\mathcal{D}$ is inner, therefore, there exists $h = y + e \in \mathfrak{h}(\mathfrak{g})$ such that $\mathcal{D} = \mathrm{ad}_h$. For any $x \in \mathfrak{g}$, since $\mathcal{D}|_{\mathfrak{g}} = 0$, we have

$$\begin{aligned}0 &= \mathcal{D}(x) \\ &= [h, x] \\ &= [y, x] + e(x),\end{aligned} \tag{3.34}$$

i.e., $e = -\mathrm{ad}_y$. Then by (3.30),

$$\begin{aligned}x_d - \mathrm{ad}_{x_d} &= \mathcal{D}(d) \\ &= [h, d] \\ &= -d(y) - [\mathrm{ad}_y, d] \\ &= -d(y) + \mathrm{ad}_{d(y)}.\end{aligned} \tag{3.35}$$

Hence $\mathrm{ad}_{x_d} = -\mathrm{ad}_{d(y)}$. Then for any $d \in \mathrm{Der}\,\mathfrak{g}$,

$$\begin{aligned}[D, d] &= \mathrm{ad}_{x_d} \\ &= -\mathrm{ad}_{d(y)} \\ &= [\mathrm{ad}_y, d].\end{aligned} \tag{3.36}$$

Since $\mathrm{Der}\,\mathfrak{g}$ has zero center, (3.36) implies that $D = \mathrm{ad}_y \in \mathrm{ad}_{\mathfrak{g}}$, a contradiction with (3.27). Thus $C(\mathrm{Der}\,\mathfrak{g}/\mathrm{ad}_{\mathfrak{g}}) = 0$, and the proof of Theorem 3.1(ii) is complete.

□

Example 3.2. An $n \times n$ matrix $q = (q_{ij})$ over a field $\mathbb{F}$ of characteristic 0 such that

$$q_{ii} = 1 \quad \text{and} \quad q_{ji} = q_{ij}^{-1}, \tag{3.37}$$

is called a *quantum matrix*. The *quantum torus*

$$\mathbb{F}_q = \mathbb{F}_q[t_1^{\pm}, \cdots, t_n^{\pm}], \tag{3.38}$$

determined by a quantum matrix q is defined with $2n$ generators $t_1^{\pm}, \cdots, t_n^{\pm}$, and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad \text{and} \quad t_j t_i = q_{ij} t_i t_j, \tag{3.39}$$

for all $1 \leq i, j \leq n$.

For any $a = (a_1, \cdots, a_n) \in \mathbb{Z}^n$, we denote $t^a = t_1^{a_1} \cdots t_n^{a_n}$. For any $a, b \in \mathbb{Z}^n$, we define

$$\sigma(a, b) = \prod_{1 \leq i,j \leq n} q_{j,i}^{a_j b_i} \quad \text{and} \quad f(a, b) = \prod_{i,j=1}^{n} q_{j,i}^{a_j b_i}. \tag{3.40}$$

Then we have

$$t^a t^b = \sigma(a, b) t^{a+b}, \quad t^a t^b = f(a, b) t^b t^a \quad \text{and} \quad f(a, b) = \sigma(a, b)\sigma(b, a)^{-1}. \tag{3.41}$$

Note that the commutator of monomials in $\mathbb{F}_q$ satisfies

$$[t^a, t^b] = (\sigma(a, b) - \sigma(b, a))t^{a+b} = \sigma(b, a)(f(a, b) - 1)t^{a+b}, \tag{3.42}$$

for all $a, b \in \mathbb{Z}^n$. Define the *radical* of f, denoted by $\mathrm{rad}(f)$, by

$$\mathrm{rad}(f) = \{a \in \mathbb{Z}^n \mid f(a, b) = 1 \ \text{ for all } \ b \in \mathbb{Z}^n\}. \tag{3.43}$$

It is clear from (3.40) that $\mathrm{rad}(f)$ is a subgroup of $\mathbb{Z}^n$. The center of $\mathbb{F}_q$ is $C(\mathbb{F}_q) = \sum_{a \in \mathrm{rad}(f)} \mathbb{F} t^a$ and the Lie ring $\mathbb{F}_q$ has the ideals decomposition

$$\mathbb{F}_q = C(\mathbb{F}_q) \oplus [\mathbb{F}_q, \mathbb{F}_q]. \tag{3.44}$$

By (3.44), we obtain that the Lie ring $[\mathbb{F}_q, \mathbb{F}_q] \cong \mathbb{F}_q / C(\mathbb{F}_q)$ is perfect and has zero center. From Theorem 1.1, we obtain the following result.

Corollary 3.3. *Let $\mathbb{F}_q$ be a quantum torus as above. Then the derivation ring* $\mathrm{Der}([\mathbb{F}_q, \mathbb{F}_q])$ *of* $[\mathbb{F}_q, \mathbb{F}_q]$ *is complete.*

Example 3.4. A Lie ring $\mathfrak{g}$ is called a *symmetric self-dual Lie ring* if $\mathfrak{g}$ is endowed with a nondegenerate invariant symmetric inner product B. For any subspace V of $\mathfrak{g}$, we define $V^\perp = \{x \in \mathfrak{g} \mid B(x, y) = 0, \forall\, y \in V\}$. Then we can easily check that $[\mathfrak{g}, \mathfrak{g}]^\perp = C(\mathfrak{g})$, where $C(\mathfrak{g})$ is the center of $\mathfrak{g}$. Then we have

Corollary 3.5. *Assume that $\mathfrak{g}$ is a symmetric self-dual Lie ring with zero center. Then the Lie ring* $\mathrm{Der}\,\mathfrak{g}$ *is complete.*

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References

[1] Benayadi, S., The perfect Lie algebras without center and outer derivations, Annals de la Faculte des Sciences de Toulouse, (1996), 203-231.

[2] Khukhro, E., Nilpotent Group and their automorphism, de Gruyter Exposition in Mathematics, Vol 8, Berlin: Walter de Gruyter, 1994.

[3] Khukhro, E., p-Automorphisms of Finite p-Groups, London Mathematical Society Lecture Note Series, Vol 246, Cambridge: Cambridge University Press, 1998.

[4] Liao, J., Liu, H., A Krull-Schmit theorem for Lie ring, Acta Math Sci, 29A:2, (2009), 399-405.

[5] Liao, J., Zheng, G., Liu, H., Towers of derivation for Lie rings and some results on complete Lie rings, Acta Math Sci, 30B:5, (2010), 1769-1775.

[6] Meng, D., Some results on complete Lie algebras, Comm. Algebra, 22, (1994), 5457-5507.

[7] Meng, D., Wang, S., On the construction of complete Lie algebras, J. Algebra, 176, (1995), 621-637.

[8] Robinson, D., A course in the Theory of Groups, Graduate Texts in Mathematics, Vol 80, 2nd ed, New York: Springer-Verlag, 1996.

[9] Su, Y., Zhu, L., Derivation Algebras of Centerless Perfect Lie Algebras Are Complete, J. Algebra, 285, (2005), 508-515.

[10] Zhu, L., Meng, D., A class of Lie algebras with a symmetric invariant non-degenerate bilinear form, Acta Math. Sinica, 43, (2000), 1119-1126.

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