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The quasiderivations of Hom-Lie color algebras

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Abstract

In this paper, we give some basic properties of a Hom-Lie color algebra L. In particular, we prove that the quasiderivations of L can be embedded as derivations in a lager Hom-Lie color algerbra, and obtain a direct sum decomposition of $\mathrm{Der}(L)$ when the annihilator of L is equal to zero.

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1 Introduction

Hom-Lie algebras are a generalization of Lie algebras, Hom-Lie algebras are also related to deformed vector fields, the various versions of the Yang-Baxter equations, braid group representations, and quantum groups [5]. More applications of the Hom-Lie algebras, Hom-algebras can be found in [4, 6].The purpose of this paper is to generalize some beautiful results to the Quasiderivations of a Hom-Lie color algebra.

2 Preliminary Notes

Throughout this paper $\mathbf{K}$ is a field, A vector space V is Γ-graded.

Definition 2.1 *[1] Let* $\mathbf{K}$ *and* Γ *be an abelian group,A map* $\Gamma \times \Gamma \to \mathbf{K}^*$ *is called a skew-symmetric bi-character on* Γ *if the following identities hold,for all x,y,z,in* Γ

(1) $\varepsilon(x, y)\varepsilon(y, x) = 1,$

(2) $\varepsilon(x, y + z) = \varepsilon(x, y)\varepsilon(x, z),$

(3) $\varepsilon(x+y,z)=\varepsilon(x,z)\varepsilon(y,z)$,

Definition 2.2 *[1] A Hom-Lie color algebra is a quadruple* $(L,[\cdot,\cdot],\varepsilon,\alpha)$ *consisting of a* Γ*-graded vector space* L*, a bi-character* ε*, an even bilinear mapping* $[\cdot,\cdot]: L\times L\to L$ *(i.e.* $[L_\theta,L_\mu]\subseteq L_{\theta+\mu}$ *for all* $\theta,\mu\in\Gamma$*) and an even homomorphism* $\alpha: L\to L$ *such that for homogeneous elements* $x,y,z\in L$ *we have*

(1) $[x,y]=-\varepsilon(x,y)[y,x]$,

(2) $\varepsilon(z,x)[\alpha(x),[y,z]]+\varepsilon(x,y)[\alpha(y),[z,x]]+\varepsilon(y,z)[\alpha(z),[x,y]]=0.$

Definition 2.3 *[3] Let* $(L,[\cdot,\cdot],\alpha)$ *be a Hom-Lie color algebra and define the following subvector space* $\mho$ *of* $\mathrm{End}(L)$ *consisting of even linear maps* u *on* L *as follows:*

$$\mho=\{u\in\mathrm{End}(L)|\ u\alpha=\alpha u\}$$

and $\sigma:\mho\to\mho;\ \sigma(u)=\alpha u$*. Then* $\mho$ *is a Hom-Lie color algebra over* $\mathbf{K}$ *with the bracket*

$$[D_\theta,D_\mu]=D_\theta D_\mu-\varepsilon(\theta,\mu)D_\mu D_\theta$$

for all $D_\theta,D_\mu\in\mathrm{hg}(\mho)$.

Definition 2.4 *[3] Let* $(L,[\cdot,\cdot],\alpha)$ *be a multiplicative Hom-Lie color algebra. A homogeneous bilinear map* $D: L\to L$ *is said to be an* α^k*-derivation of* L*, where* $k\in\mathbf{N}$*, if it satisfies*

$$D\alpha=\alpha D,$$

$$[D(x),\alpha^k(y)]+\varepsilon(D,x)[\alpha^k(x),D(y)]=D([x,y]),$$

$\forall x\in\mathrm{hg}(L),\ y\in L$.

We denote the set of all α^k*-derivations by* $\mathrm{Der}_{\alpha^k}(L)$*, then* $\mathrm{Der}(L):=\bigoplus_{k\geq 0}\mathrm{Der}_{\alpha^k}(L)$ *provided with the super-commutator and the following even map*

$$\tilde{\alpha}:\mathrm{Der(L)}\to\mathrm{Der(L)};\quad\tilde{\alpha}(\mathrm{D})=\mathrm{D}\alpha$$

is a Hom-subalgebra of $\mho$ *and is called the derivation algebra of* L*.*

Definition 2.5 *[1] An endomorphism* $D\in\mathrm{hg}(Der(L))$ *is said to be a homogeneous* α^k*-quasiderivation , if there exists an endomorphism* $D'\in hg(\mathrm{End}(L))$ *such that*

$$D\alpha=\alpha D, D\alpha'=\alpha' D$$

$$[D(x),\alpha^k(y)]+\varepsilon(D,x)[\alpha^k(x),D(y)]=D'([x,y]),\tag{1.1}$$

for all $x\in\mathrm{hg}(L),y\in L$*. Let* $\mathrm{QDer}_{\alpha^k}(L)$ *be the sets of homogeneous* α^k*-quasiderivations.*

Definition 2.6 *[3] Let* $(L,[\cdot,\cdot],\alpha)$ *be a multiplicative Hom-Lie color algebra. If* $\mathrm{Z}(L):=\bigoplus_{\theta\in\Gamma}\mathrm{Z}_\theta(L)$*, with* $\mathrm{Z}_\theta(L)=\{x\in L_\theta|[x,y]=0,\forall x\in\mathrm{hg}(L),y\in L\}$*, then* $\mathrm{Z}(L)$ *is called the center of* L*.*

3 Main Results

Lemma 3.1 *Let $(L, [\cdot,\cdot], \alpha)$ be a Hom-Lie color algebra over* $\mathbf{K}$ *and t an indeterminate. We define $\breve{L}_g := L_g[tF[t]/(t^3)] = \{\Sigma(x_g \otimes t + y_g \otimes t^2) | x_g, y_g \in L_g\}$, $\breve{\alpha}(\breve{L}_g) := \{\Sigma(\alpha(x_g) \otimes t + \alpha(y_g) \otimes t^2) : x_g, y_g \in L_g\}$, and let $\breve{L} = \breve{L}_{\bar{0}} \oplus \breve{L}_{\bar{1}}$. Then $\breve{L}$ is a Hom-Lie color algebra with the operation $[x_\lambda \otimes t^i, x_\theta \otimes t^j] = [x_\lambda, x_\theta] \otimes t^{i+j}$, for all $x_\lambda, x_\theta \in \mathrm{hg}(L), i, j \in \{1, 2\}$.*

Proof. *For all $x_\lambda, x_\theta, x_\mu \in \mathrm{hg}(L)$ and $i, j, k \in \{1, 2\}$, we have*

$$\begin{aligned}[x_\lambda \otimes t^i, x_\theta \otimes t^j] &= [x_\lambda, x_\theta] \otimes t^{i+j} \\ &= -\varepsilon(\lambda, \theta)[x_\theta \otimes t^j, x_\lambda \otimes t^i],\end{aligned}$$

$$\begin{aligned}&[\breve{\alpha}(x_\lambda \otimes t^i), [x_\theta \otimes t^j, x_\mu \otimes t^k]] = [\alpha(x_\lambda), [x_\theta, x_\mu]] \otimes t^{i+j+k} \\ &= ([[x_\lambda, x_\theta], \alpha(x_\mu)] + \varepsilon(\lambda, \theta)[\alpha(x_\theta), [x_\lambda, x_\mu]]) \otimes t^{i+j+k} \\ &= [[x_\lambda \otimes t^i, x_\theta \otimes t^j], \breve{\alpha}(x_\mu \otimes t^k)] + \varepsilon(\lambda, \theta)[\breve{\alpha}(x_\theta \otimes t^j), [x_\lambda \otimes t^i, x_\mu \otimes t^k]].\end{aligned}$$

Hence $\breve{L}$ is a Hom-Lie color algebra. □

For notational convenience, we write $xt(xt^2)$ in place of $x \otimes t(x \otimes t^2)$. If U is a Γ-graded subspace of L such that $L = U \oplus [L, L]$, then $\breve{L} = Lt + Lt^2 = Lt + [L, L]t^2 + Ut^2$,

Now we define a map $\varphi : \mathrm{QDer}(L) \to \mathrm{End}(\breve{L})$ satisfying $\varphi(D)(at + bt^2 + ut^2) = D(a)t + D'(b)t^2$, where D, D' satisfy (1.1), $a \in \mathrm{hg}(L), b \in \mathrm{hg}([L, L]), u \in \mathrm{hg}(U)$ and $d(a) = d(b) = d(u)$.

Lemma 3.2 *(1)* $d(\varphi) = 0$.

(2) φ is injective and $\varphi(D)$ does not depend on the choice of D'.

(3) $\varphi(\mathrm{QDer}(L)) \subseteq \mathrm{Der}(\breve{L})$.

Proof. It is clear.

(2) If $\varphi(D_\lambda) = \varphi(D_\theta)$, then for all $a \in \mathrm{hg}(L), b \in \mathrm{hg}([L, L])$ and $u \in \mathrm{hg}(U)$, we have

$$D_\lambda(a)t + D'_\lambda(b)t^2 = D_\theta(a)t + D'_\theta(b)t^2,$$

so $D_\lambda(a) = D_\theta(a)$. Hence $D_\lambda = D_\theta$, and φ is injective.

Suppose that there exists D'' such that

$$\varphi(D)(at + bt^2 + ut^2) = D(a)t + D''(b)t^2,$$

and

$$[D(x), \alpha^k(y)] + \varepsilon(D, x)[\alpha^k(x), D(y)] = D''([x, y]),$$

then we have

$$D'([x,y]) = D''([x,y]),$$

thus $D'(b) = D''(b)$. Hence

$$\varphi(D)(at + bt^2 + ut^2) = D(a)t + D'(b)t^2 = D(a)t + D''(b)t^2,$$

which implies $\varphi(D)$ is determined by D.

(3) We have $[x_\lambda t^i, x_\theta t^j] = [x_\lambda, x_\theta]t^{i+j} = 0$, for all $i + j \geq 3$. Thus, to show $\varphi(D) \in \mathrm{Der}(\breve{L})$, we need only to check the validness of the following equation

$$\varphi(D)([xt, yt]) = [\varphi(D)(xt), \breve{\alpha}^k(yt)] + \varepsilon(D, x)[\breve{\alpha}^k(xt), \varphi(D)(yt)].$$

For all $x, y \in \mathrm{hg}(L)$, we have

$$\begin{aligned}\varphi(D)([xt, yt]) &= \varphi(D)([x,y]t^2) = D'([x,y])t^2 \\ &= [\varphi(D)(xt), \breve{\alpha}^k(yt)] + \varepsilon(D, x)[\breve{\alpha}^k(xt), \varphi(D)(yt)].\end{aligned}$$

Therefore, for all $D \in \mathrm{QDer}(L)$, we have $\varphi(D) \in \mathrm{Der}(\breve{L})$

□

Lemma 3.3 *Let $(L, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie color algebra and α a surjection. $\mathrm{Z}(L) = \{0\}$ and $\breve{L}$, φ are as defined above. Then $\mathrm{Der}(\breve{L}) = \varphi(\mathrm{QDer}(L)) \oplus \mathrm{ZDer}(\breve{L})$. Proof. Since $\mathrm{Z}(L) = \{0\}$, we have $\mathrm{Z}(\breve{L}) = Lt^2$. For all $g \in \mathrm{Der}(\breve{L})$, we have $g(\mathrm{Z}(\breve{L})) \subseteq \mathrm{Z}(\breve{L})$, hence $g(Ut^2) \subseteq g(\mathrm{Z}(\breve{L})) \subseteq \mathrm{Z}(\breve{L}) = Lt^2$. Now we define a map $f : Lt + [L,L]t^2 + Ut^2 \to Lt^2$ by*

$$f(x) = \begin{cases} g(x) \cap Lt^2, & x \in Lt; \\ g(x), & x \in Ut^2; \\ 0, & x \in [L,L]t^2. \end{cases}$$

Proof. It is clear that f is linear. Note that

$$f([\breve{L}, \breve{L}]) = f([L,L]t^2) = 0,\ [f(\breve{L}), \breve{\alpha}^k L] \subseteq [Lt^2, \alpha^k(L)t + \alpha^k(L)t^2] = 0,$$

hence $f \in \mathrm{ZDer}(\breve{L})$. Since

$$(g - f)(Lt) = g(Lt) - g(Lt) \cap Lt^2 = g(Lt) - Lt^2 \subseteq Lt,\ (g - f)(Ut^2) = 0,$$

and

$$(g - f)([L,L]t^2) = g([\breve{L}, \breve{L}]) \subseteq [\breve{L}, \breve{L}] = [L,L]t^2,$$

there exist $D,\ D' \in \mathrm{End}(L)$ such that for all $a \in L,\ b \in [L,L]$,

$$(g - f)(at) = D(a)t,\ (g - f)(bt^2) = D'(b)t^2.$$

Since $(g - f) \in \mathrm{Der}(\breve{L})$ *and by the definition of* $\mathrm{Der}(\breve{L})$*, we have*

$$[(g - f)(a_1 t), \breve{\alpha}^k(a_2 t)] + \varepsilon(g - f, a_1)[\breve{\alpha}^k(a_1 t), (g - f)(a_2 t)] = (g - f)([a_1 t, a_2 t]),$$

for all $a_1, a_2 \in L$*. Hence*

$$[D(a_1), \breve{\alpha}^k(a_2)] + \varepsilon(D, a_1)[\breve{\alpha}^k(a_1), D(a_2)] = D'([a_1, a_2]).$$

Thus $D \in \mathrm{QDer}(L)$*. Therefore,*

$$g - f = \varphi(D) \in \varphi(\mathrm{QDer}(L)) \Rightarrow \mathrm{Der}(\breve{L}) \subseteq \varphi(\mathrm{QDer}(L)) + \mathrm{ZDer}(\breve{L}).$$

By Lemma 3.2 (3) *we have*

$$\mathrm{Der}(\breve{L}) = \varphi(\mathrm{QDer}(L)) + \mathrm{ZDer}(\breve{L}).$$

For all $f \in \varphi(\mathrm{QDer}(L)) \cap \mathrm{ZDer}(\breve{L})$*, there exists an element* $D \in \mathrm{QDer}(L)$ *such that* $f = \varphi(D)$*. Then*

$$f(at + bt^2 + ut^2) = \varphi(D)(at + bt^2 + ut^2) = D(a)t + D'(b)t^2,$$

for all $a \in L, b \in [L, L]$.

On the other hand, $D(a) = 0$*, for all* $a \in L$ *and so* $D = 0$*. Hence* $f = 0$.

Therefore $\mathrm{Der}(\breve{L}) = \varphi(\mathrm{QDer}(L)) \oplus \mathrm{ZDer}(\breve{L})$ *as desired.* □

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