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Hyers-Ulam Stability of A Kind of Polynomial Equation

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Abstract

In this paper, we prove the stability in the sense of Hyers-Ulam stability of a kind of polynomial equation. That is, if y is an approximate solution of the polynomial equation $a_ny^n+a_{n-1}y^{n-1}+\cdots+a_1y+a_0=0$, then there exists an exact solution of the polynomial equation near to y.

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1 Introduction

In 1940, S. M. Ulam [1] gave a wide ranging talk about the important concerning the stability of homomorphisms. In 1941, Hyers [2] solved the problem for the case of approximately additive mappings between spaces. Since then, the stability problems of functional equations have been extensively studied by several authors $[3-7]$.

Y. Li remarked that the polynomial equation $x^n+\alpha x+\beta=0$ and the differential equation $y''=\lambda y$, $y''+\alpha y'(t)+\beta y=f(t)$ have the Hyers-Ulam Stability $[8-9]$. Recently, Jung studied the Hyers-Ulam Stability of several types of linear differential equations of second order $[10-15]$.

Motivated by and connected to the results mentioned above and [8], we consider stability problems for a polynomial equation. In this paper, we will studied the Hyers-Ulam Stability of the following polynomial equation

$$x^n+\alpha x^2+\beta x+\gamma=0. \tag{1}$$

where $x \in [-1, 1]$.

We say that equation (1) has the Hyers-Ulam Stability if there exists a constant $K > 0$ with the following property: for every $\varepsilon > 0$, $y \in [-1, 1]$, if

$$y^n + \alpha y^2 + \beta y + \gamma \leq \varepsilon$$

then there exists some $z \in [-1, 1]$ satisfying

$$z^n + \alpha z^2 + \beta z + \gamma = 0$$

such that $|y - z| < K\varepsilon$. K is the Hyers-Ulam Stability constant for equation (1).

The similar conclusions apply to the following equation

$$a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0 = 0. \tag{2}$$

2 Main Results

Now, the main result of this work is given in the following theorem.

Theorem 2.1. *If $2|\alpha| + n \leq |\beta|$ and $y \in [-1, 1]$ satisfies the inequality*

$$|y^n + \alpha y^2 + \beta y + \gamma| \leq \varepsilon$$

then there exists a solution $z \in [-1, 1]$ of Eq.(1) such that

$$|y - z| \leq K\varepsilon$$

where $K > 0$ is a constant.

Proof. Let $\varepsilon > 0$ and $y \in [-1, 1]$ such that

$$|y^n + \alpha y^2 + \beta y + \gamma| \leq \varepsilon$$

We need to prove that there exists a constant K independent of ε and z such that $|y - z| \leq K\varepsilon$ for some $z \in [-1, 1]$ satisfying $x^n + \alpha x^2 + \beta x + \gamma = 0$.

Let

$$T(x) = \frac{1}{\beta}(-x^n - \alpha x^2 - \gamma) \qquad x \in [-1, 1]$$

then

$$|T(x)| = |\frac{1}{\beta}(-x^n - \alpha x^2 - \gamma)| \leq 1.$$

Let $X = [-1, 1]$, $d(x, y) = |x - y|$, then (X, d) is a complete metric space, and T map X into X.

We will prove that T is a contraction mapping from X to X. For any $x, y \in X$, one has

$$\begin{aligned} d(T(x), T(y)) &= |\frac{1}{\beta}(-x^n - \alpha x^2 - \gamma) - \frac{1}{\beta}(-y^n - \alpha y^2 - \gamma)| \\ &\leq \frac{1}{|\beta|}|(x^n - y^n) + \alpha(x^2 - y^2)| \\ &\leq \frac{1}{|\beta|}(|x-y||x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}| + |\alpha||x-y||x+y|) \end{aligned}$$

Since $x, y \in [-1, 1], x \neq y, 2|\alpha| + n \leq |\beta|$, we obtain

$$d(T(x), T(y)) \leq Cd(x, y)$$

where $C = \frac{2|\alpha|+n}{|\beta|} \in [0, 1]$.

Thus, T is a contraction mapping from X to X, by S.Banachs contraction mapping theorem, there exists unique $z \in X$, such that

$$T(z) = z$$

Hence, Eq.(1) has a solution on $[-1, 1]$.

Finally, Eq.(1) of the Hyers-Ulam stability will be showed.

$$|y - z| = |y - T(y) + T(y) - T(z)| \leq |y - \frac{1}{\beta}(-y^n - \alpha y^2 - \gamma - \beta y)| + C|y - z|$$

Thus, we get

$$|y - z| \leq \frac{1}{|\beta|(1 - C)}|y^n + \alpha y^2 + \gamma + \beta y)|$$

Let $\frac{1}{|\beta|(1-C)} = K$, we get

$$|y - z| \leq K|y^n + \alpha y^2 + \gamma + \beta y)| \leq K\varepsilon.$$

Proof of Theorem 2.1 is complete. □

By applying a similar argument of the proof of Theorem 2.1, it is easy to see the following theorem holds.

Theorem 2.2. *If* $|na_n + (n-1)a_{n-1} + \cdots + 2a_2| \leq |a_1|$ *and* $y \in [-1, 1]$ *satisfies the inequality*

$$|a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0| \leq \varepsilon$$

then there exists a solution $z \in [-1, 1]$ *of Eq.(2) such that*

$$|y - z| \leq K\varepsilon$$

where $K > 0$ *is a constant.*

Proof. Let $\varepsilon > 0$ and $y \in [-1,1]$ such that

$$|a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0| \leq \varepsilon$$

We need to prove that there exists a constant K independent of ε and z such that $|y - z| \leq K\varepsilon$ for some $z \in [-1,1]$ satisfying Eq.(2).

Let

$$T(x) = \frac{1}{a_1}(-a_n x^n - a_{n-1} x^{n-1} - \cdots - a_2 x^2 - a_0) \qquad x \in [-1,1]$$

then

$$|T(x)| = |\frac{1}{\beta}(-x^n - \alpha x^2 - \gamma)| \leq 1.$$

Let $X = [-1,1]$, $d(x,y) = |x - y|$, then (X,d) is a complete metric space, and T map X into X.

We will prove that T is a contraction mapping from X to X. For any $x, y \in X$, one has

$$\begin{aligned} d(T(x),T(y)) = &|\frac{1}{a_1}(-a_n x^n - a_{n-1} x^{n-1} - \cdots - a_2 x^2 - a_0) \\ &- \frac{1}{a_1}(-a_n y^n - a_{n-1} y^{n-1} - \cdots - a_2 y^2 - a_0)| \\ \leq &\frac{1}{|a_1|}|a_n(x^n - y^n) + a_{n-1}(x^{n-1} - y^{n-1}) + \cdots + a_2(x^2 - y^2)| \\ \leq &\frac{1}{|a_1|}(|x - y||a_n(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) + \cdots + a_2(x + y)| \end{aligned}$$

Since $x, y \in [-1,1], x \neq y, |na_n + (n-1)a_{n-1} + \cdots + 2a_2| \leq |a_1|$, we obtain

$$d(T(x),T(y)) \leq C d(x,y)$$

where $C = \frac{|na_n + (n-1)a_{n-1} + \cdots + 2a_2}{|a_1|} \in [0,1]$.

Thus, T is a contraction mapping from X to X, by S.Banachs contraction mapping theorem, there exists unique $z \in X$, such that

$$T(z) = z$$

Hence, Eq.(2) has a solution on $[-1,1]$.

Finally, Eq.(2) of the Hyers-Ulam stability will be showed.

$$\begin{aligned} |y - z| &= |y - T(y) + T(y) - T(z)| \leq |y - T(y)| + |T(y) - T(z)| \\ &\leq |y - \frac{1}{a_1}(-a_n y^n - a_{n-1} y^{n-1} - a_2 y^2 - a_0)| + C|y - z| \end{aligned}$$

Thus, we get

$$|y - z| \leq \frac{1}{|a_1|(1 - C)}|a_n y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0|$$

Let $\frac{1}{|a_1|(1-C)} = K$, we get

$$|y - z| \leq K\varepsilon.$$

Proof of Theorem 2.2 is complete. □

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