Hyers-Ulam Stability of A Kind of Polynomial Equation

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Abstract

In this paper, we prove the stability in the sense of Hyers-Ulam stability of a kind of polynomial equation. That is, if \( y \) is an approximate solution of the polynomial equation

\[
a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0 = 0,
\]

then there exists an exact solution of the polynomial equation near to \( y \).

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1 Introduction

In 1940, S. M. Ulam [1] gave a wide ranging talk about the important concerning the stability of homomorphisms. In 1941, Hyers [2] solved the problem for the case of approximately additive mappings between spaces. Since then, the stability problems of functional equations have been extensively studied by several authors [3 – 7].

Y. Li remarked that the polynomial equation \( x^n + \alpha x + \beta = 0 \) and the differential equation \( y'' = \lambda y, \ y'' + \alpha y'(t) + \beta y = f(t) \) have the Hyers-Ulam Stability [8 – 9]. Recently, Jung studied the Hyers-Ulam Stability of several types of linear differential equations of second order [10 – 15].

Motivated by and connected to the results mentioned above and [8], we consider stability problems for a polynomial equation. In this paper, we will studied the Hyers-Ulam Stability of the following polynomial equation

\[
x^n + \alpha x^2 + \beta x + \gamma = 0. \tag{1}
\]
where \( x \in [-1, 1] \).

We say that equation (1) has the Hyers-Ulam Stability if there exists a constant \( K > 0 \) with the following property: for every \( \varepsilon > 0 \), \( y \in [-1, 1] \), if

\[
y^n + \alpha y^2 + \beta y + \gamma \leq \varepsilon
\]

then there exists some \( z \in [-1, 1] \) satisfying

\[
z^n + \alpha z^2 + \beta z + \gamma = 0
\]

such that \(|y - z| < K\varepsilon\). \( K \) is the Hyers-Ulam Stability constant for equation (1).

The similar conclusions apply to the following equation

\[
a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0 = 0.
\]

(2)

2 Main Results

Now, the main result of this work is given in the following theorem.

Theorem 2.1. If \( 2|\alpha| + n \leq |\beta| \) and \( y \in [-1, 1] \) satisfies the inequality

\[
|y^n + \alpha y^2 + \beta y + \gamma| \leq \varepsilon
\]

then there exists a solution \( z \in [-1, 1] \) of Eq.(1) such that

\[
|y - z| \leq K\varepsilon
\]

where \( K > 0 \) is a constant.

Proof. Let \( \varepsilon > 0 \) and \( y \in [-1, 1] \) such that

\[
|y^n + \alpha y^2 + \beta y + \gamma| \leq \varepsilon
\]

We need to prove that there exists a constant \( K \) independent of \( \varepsilon \) and \( z \) such that \(|y - z| \leq K\varepsilon\) for some \( z \in [-1, 1] \) satisfying \( x^n + \alpha x^2 + \beta x + \gamma = 0 \).

Let

\[
T(x) = \frac{1}{\beta}(-x^n - \alpha x^2 - \gamma) \quad x \in [-1, 1]
\]

then

\[
|T(x)| = \frac{1}{\beta}|-x^n - \alpha x^2 - \gamma| \leq 1.
\]

Let \( X = [-1, 1] \), \( d(x, y) = |x - y| \), then \((X, d)\) is a complete metric space, and \( T \) map \( X \) into \( X \).
We will prove that $T$ is a contraction mapping from $X$ to $X$. For any $x, y \in X$, one has
\[
d(T(x), T(y)) = \left| \frac{1}{\beta}(-x^n - \alpha x^2 - \gamma) - \frac{1}{\beta}(-y^n - \alpha y^2 - \gamma) \right|
\leq \frac{1}{|\beta|} \left| (x^n - y^n) + \alpha (x^2 - y^2) \right|
\leq \frac{1}{|\beta|} \left| (x - y) |x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}| + |\alpha| |x - y||x + y| \right|
\]

Since $x, y \in [-1, 1], x \neq y, 2|\alpha| + n \leq |\beta|$, we obtain
\[
d(T(x), T(y)) \leq Cd(x, y)
\]
where $C = \frac{2|\alpha| + n}{|\beta|} \in [0, 1]$.

Thus, $T$ is a contraction mapping from $X$ to $X$, by S.Banachs contraction mapping theorem, there exists unique $z \in X$, such that
\[
T(z) = z
\]
Hence, Eq.(1) has a solution on $[-1, 1]$.

Finally, Eq.(1) of the Hyers-Ulam stability will be showed.

\[
|y - z| = |y - T(y) + T(y) - T(z)| \leq |y - \frac{1}{\beta}(-y^n - \alpha y^2 - \gamma - \beta y)| + C|y - z|
\]
Thus, we get
\[
|y - z| \leq \frac{1}{|\beta|(1 - C)} |y^n + \alpha y^2 + \gamma + \beta y|
\]
Let $\frac{1}{|\beta|(1 - C)} = K$, we get
\[
|y - z| \leq K|y^n + \alpha y^2 + \gamma + \beta y| \leq K\varepsilon.
\]
Proof of Theorem 2.1 is complete. \hfill \square

By applying a similar argument of the proof of Theorem 2.1, it is easy to see the following theorem holds.

**Theorem 2.2.** If $|na_n + (n - 1)a_{n-1} + \cdots + 2a_2| \leq |a_1|$ and $y \in [-1, 1]$ satisfies the inequality
\[
|a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0| \leq \varepsilon
\]
then there exists a solution $z \in [-1, 1]$ of Eq.(2) such that
\[
|y - z| \leq K\varepsilon
\]
where $K > 0$ is a constant.
Proof. Let $\varepsilon > 0$ and $y \in [-1, 1]$ such that
$$|a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0| \leq \varepsilon$$

We need to prove that there exists a constant $K$ independent of $\varepsilon$ and $z$ such that $|y - z| \leq K\varepsilon$ for some $z \in [-1, 1]$ satisfying Eq.(2).

Let
$$T(x) = \frac{1}{a_1}(-a_n x^n - a_{n-1} x^{n-1} - \cdots - a_2 x^2 - a_0) \quad x \in [-1, 1]$$

then
$$|T(x)| = \frac{1}{|a_1|}(-x^n - \alpha x^2 - \gamma) \leq 1.$$

Let $X = [-1, 1]$, $d(x, y) = |x - y|$, then $(X, d)$ is a complete metric space, and $T$ map $X$ into $X$.

We will prove that $T$ is a contraction mapping from $X$ to $X$. For any $x, y \in X$, one has

$$d(T(x), T(y)) = \frac{1}{a_1}(-a_n x^n - a_{n-1} x^{n-1} - \cdots - a_2 x^2 - a_0)
- \frac{1}{a_1}(-a_n y^n - a_{n-1} y^{n-1} - \cdots - a_2 y^2 - a_0)
\leq \frac{1}{|a_1|}(|x^n - y^n| + a_{n-1} |x^{n-1} - y^{n-1}| + \cdots + a_2 |x^2 - y^2|)
\leq \frac{1}{|a_1|}(|x - y| |a_n x^{n-2} + a_{n-1} x^{n-3} + \cdots + y^{n-2} + y^{n-3} + \cdots + a_2 x + y|)$$

Since $x, y \in [-1, 1], x \neq y, |na_n + (n-1)a_{n-1} + \cdots + 2a_2| \leq |a_1|$, we obtain
$$d(T(x), T(y)) \leq C d(x, y)$$

where $C = \frac{|na_n + (n-1)a_{n-1} + \cdots + 2a_2|}{|a_1|} \in [0, 1]$. Thus, $T$ is a contraction mapping from $X$ to $X$, by S.Banachs contraction mapping theorem, there exists unique $z \in X$, such that
$$T(z) = z$$

Hence, Eq.(2) has a solution on $[-1, 1]$.

Finally, Eq.(2) of the Hyers-Ulam stability will be showed.

$$|y - z| = |y - T(y) + T(y) - T(z)| \leq |y - T(y)| + |T(y) - T(z)|
\leq |y - \frac{1}{a_1}(-a_n y^n - a_{n-1} y^{n-1} - a_2 y^2 - a_0)| + C|y - z|$$
Thus, we get

\[ |y - z| \leq \frac{1}{|a_1|(1 - C)} \left| a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0 \right| \]

Let \( \frac{1}{|a_1|(1 - C)} = K \), we get

\[ |y - z| \leq K \varepsilon. \]

Proof of Theorem 2.2 is complete. \( \square \)

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**References**


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