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A Class of Univalent Harmonic Mappings

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Abstract

The main aim of this paper is to investigate a class of univalent harmonic mappings, which is the generalization of the class of analytic functions whose derivative has a positive real part. The distortion theorem, the radius of convexity, univalence of the partial sums, the extremal property of $f_0(z) = -z - 2\log(1-z)$, and functions with initial zero coefficients are considered.

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1 Introduction

Assume that $f = u + iv$ is a complex-valued harmonic function defined in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, i.e. u and v are real harmonic in $\mathbb{D}$. Then f admits the decomposition $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$, see [4]. Often h and g are referred to as the analytic and co-analytic parts of f, respectively. If in addition f is univalent in $\mathbb{D}$, then f has a non-vanishing Jacobian in $\mathbb{D}$, where the Jacobian of f is given by

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2.$$

We say that f is sense-preserving in $\mathbb{D}$ if $J_f(z) > 0$ in $\mathbb{D}$. Moreover, the converse is also true, see [6]. If f is sense preserving, then the complex dilatation $\omega := g'/h'$ is analytic in $\mathbb{D}$ and maps $\mathbb{D}$ into $\mathbb{D}$.

The condition $Ref'(z) > 0$ is known to be sufficient for the univalence of an analytic function f in any convex domain, see [3, Theorem 2.16]. An early consideration of functions satisfying the condition $Ref'(z) > 0$ can be found in the paper by Alexander [1]. He proves: if f is analytic in $\mathbb{D}$ and f' "maps $\mathbb{D}$ upon a region contained within a half-plane bounded by a straight line through the origin", then f is univalent. Wolff [13] showed that f is univalent in $Rez > 0$ if it is analytic there and satisfies $Ref'(z) > 0$. Noshiro [8, P_{151}] and Warschawski [12, P_{312}] each demonstrated that $Ref'(z) > 0$ is a sufficient condition for the univalence of f in any convex domain. Conversely, Tims [11] proved that for each simply connected nonconvex domain D there is a function f analytic in D such that $Ref'(z) > 0$ and f is not univalent in D. A more general class of functions than those satisfying $Ref'(z) > 0$ is the class of close-to-convex functions, which is univalent and the range is close-to-convex, see, for example, [3]. Let $\mathcal{R}$ denote the class of analytic functions which satisfies $Ref'(z) > 0$ in $\mathbb{D}$ and is normalized by $f(0) = 0$ and $f'(0) = 1$. Because of Alexander's result each function in $\mathcal{R}$ is univalent, see [1]. In [7], author continued to investigate functions in $\mathcal{R}$, and considered distortion theorems, the radius of convexity, univalence of the partial sums, further extremal properties of $f_0(z) = -z - 2\log(1 - z)$, and functions with initial zero coefficients. The aim of this paper is to generalize the results in [7] to the case of harmonic mappings.

A natural generalization of the class $\mathcal{R}$ to the case of harmonic mappings is the class $\mathcal{H}$ of harmonic mappings $f = h + \overline{g}$, defined by the condition $Reh'(z) > |g'(z)|$ and satisfied $h(0) = h'(0) - 1 = 0$, $g(0) = g'(0) = 0$. Clearly, if $f = h + \overline{g} \in \mathcal{H}$, then $h \in \mathcal{R}$ and $h + e^{i\theta} g \in \mathcal{R}$ for each $\theta \in [0, 2\pi)$. This fact plays an important role in the proof of our main results. Harmonic mappings in $\mathcal{H}$ have many interesting properties see [2, 5, 9]. In particular, the authors [9] have shown that $f \in \mathcal{H}$ is indeed close to convex in $\mathbb{D}$.

In this paper, we continue to consider the harmonic mappings in $\mathcal{H}$, and generalize the properties of functions in $\mathcal{R}$ to the case of harmonic mappings in $\mathcal{H}$. By using different methods, we discuss the distortion theorem, the radius of convexity, univalence of the partial sums, extremal property of $f_0(z) = -z - 2\log(1 - z)$, and functions with initial zero coefficients, which are generalizations of the corresponding results in [7] to the case of harmonic mappings.

2 Distortion theorem

In [9], the following result has been proved.

Theorem A Suppose that $f = h + \overline{g}$, where $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = \sum_{k=2}^{\infty} b_k z^k$ in a neighborhood of the origin and $|b_1| < 1$. If

$$\sum_{k=2}^{\infty} k|a_k| + \sum_{k=2}^{\infty} k|b_k| \leq 1,$$

then $f \in \mathcal{H}$.

In fact, we have the following sharp estimate of coefficients for $f \in \mathcal{H}$.

Theorem 2.1 *If $f(z) = h(z) + \overline{g(z)} = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{\infty} \overline{b_k}\overline{z}^k \in \mathcal{H}$, then*

$$|a_k| + |b_k| \leq \frac{2}{k} \text{ for } k = 2, 3, \cdots, \quad |h'(z)| + |g'(z)| \leq \frac{1 + |z|}{1 - |z|},$$

$$Re(h'(z)) \geq \frac{1 - |z|}{1 + |z|} + |g'(z)|$$

$$-|z| + 2\log(1 + |z|) \leq ||h(z)| - |g(z)|| \leq |h(z)| + |g(z)| \leq -|z| - 2\log(1 - |z|). \tag{1}$$

Since for $f = h + \overline{g} \in \mathcal{H}$, $h + e^{i\theta} g \in \mathcal{R}$ for each $\theta \in [0, 2\pi)$, that is $Re(h'(z) + e^{i\theta} g'(z)) > 0$ in $\mathbb{D}$. By using [7, Theorem 1], Theorem 2.1 follows.

Let $f_0(z) = -z - 2\log(1 - z) = z + \sum_{n=2}^{\infty} (2/n) z^n$. By considering f_0, we can verify that all estimates of this theorem are sharp. The next results follows from estimate (1).

Corollary 2.2 *Each mapping in $\mathcal{H}$ maps $\mathbb{D}$ onto a domain which covers the disc $|\omega| < 2\log 2 - 1$.*

3 The radius of convexity

We begin this section with the following lemma.

Lemma 3.1 ([4, P_{38}, Theorem]) *Let $f = h + \overline{g}$ be harmonic and locally univalent in $\mathbb{D}$. Then f is univalent and its range is convex if and only if for each choice of α $(0 \leq \alpha < 2\pi)$, the analytic function $e^{i\alpha} h - e^{-i\alpha} g$ is univalent and its range is convex in the horizontal direction.*

Theorem 3.2 *Each mapping in $\mathcal{H}$ maps $|z| < \sqrt{2} - 1$ onto a convex domain.*

Proof Suppose that $f = h + \overline{g} \in \mathcal{H}$. Let

$$f_r(z) = \frac{1}{r}(h(rz) + \overline{g(rz)}) = h_r(z) + \overline{g_r(z)} \text{ and } f_\theta(z) = h(z) + e^{i\theta}g(z)$$

for $0 < r < 1$. Then $f_r \in \mathcal{H}$, $f_\theta \in \mathcal{R}$. For the function $f_{\theta,\frac{1}{2}}(z) := 2f_\theta(\frac{1}{2}z) = h_{\frac{1}{2}}(z)+e^{i\theta}g_{\frac{1}{2}}(z)$, by using [7, Theorem 2] shows that $f_{\theta,\frac{1}{2}}$ maps $\mathbb{D}$ onto a convex domain. Then for each $\alpha \in [0, 2\pi)$ and $\theta = \pi - 2\alpha$,

$$e^{i\alpha}f_{\theta,\frac{1}{2}}(z) = e^{i\alpha}h_{\frac{1}{2}}(z) + e^{i\alpha}e^{i\pi-2i\alpha}g_{\frac{1}{2}}(z) = e^{i\alpha}h_{\frac{1}{2}}(z) - e^{-i\alpha}g_{\frac{1}{2}}(z)$$

maps $\mathbb{D}$ onto a convex domain, it follows from Lemma 3.1 that $f_{\frac{1}{2}}$ is convex, which implies that f maps $|z| < 1/2$ onto a convex domain.

For the function $f_0(z) = -z - 2\log(1 - z)$, we have $[zf_0''(z)/f_0'(z) + 1] = (1+2z-z^2)/(1-z^2)$ and this last expression vanishes for $z = \sqrt{2}-1$. Hence, this function maps no circle $|z| < r$ larger than $|z| < \sqrt{2} - 1$ onto a convex domain.

4 Univalence of the partial sums

It is interesting to determine to what extent a given property of a power series is carried over to its partial sums. Szegö [10] has shown that all of the partial sums of a function univalent in $\mathbb{D}$ are univalent in $|z| < 1/4$. For mappings in $\mathcal{R}$, authors [7] proved that the constant 1/4 can be improved to 1/2. We generalize this result to the case of harmonic mappings.

Theorem 4.1 *Let* $f(z) = h(z) + \overline{g(z)} = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{\infty} \overline{b_k}\overline{z}^k \in \mathcal{H}$. *Then* $f_n(z) = h_n(z) + \overline{g_n(z)} = z + \sum_{k=2}^{n} a_k z^k + \sum_{k=2}^{n} \overline{b_k}\overline{z}^k$ *is univalent in* $|z| < 1/2$ *for* $n = 2, 3, \cdots$.

Proof Since $f = h + \overline{g} \in \mathcal{H}$, it follows that $f_\theta = h + e^{i\theta}g \in \mathcal{R}$. From the proof of [7, Theorem 4], we obtain that $Re(h_n'(z) + e^{i\theta}g_n'(z)) > 0$ in $|z| < 1/2$, which implies that $Reh_n'(z) > |g_n'(z)|$ in $|z| < 1/2$, and then f_n is univalent in $|z| < 1/2$.

5 Extremal property of $f_0(z) = -z - 2\log(1 - z)$

Theorem 5.1 *Suppose* $0 < r < 1$. *The area of the image of* $|z| < r$ *for mappings in* $\mathcal{H}$ *is maximal for* $f_0(z) = -z - 2\log(1 - z)$.

Proof Suppose that $f = h + \overline{g} \in \mathcal{H}$ and $0 < r < 1$. Then $h \in \mathcal{R}$. Let $A_r(f)$ (resp. $A_r(h)$, $A_r(f_0)$) be the image of $|z| < r$ under f (resp. h, f_0). By using [7, Theorem 5], it follows that

$$\begin{aligned} A_r(f) &= \int\int_{\mathbb{D}} J_f(z)dxdy = \int\int_{\mathbb{D}} (|h'(z)|^2 - |g'(z)|^2)dxdy \\ &\leq \int\int_{\mathbb{D}} |h'(z)|^2 dxdy \\ &= A_r(h) \\ &\leq A_r(f_0). \end{aligned}$$

The proof of this theorem is complete.

6 Mappings with initial zero coefficients

Some of the results obtained for mappings in $\mathcal{H}$ can be improved if f has the form

$$f(z) = z + \sum_{k=n}^{\infty} a_k z^k + \sum_{k=n}^{\infty} \overline{b_k} \overline{z}^k.$$

By using [7, Theorem 6], similar to Theorem 2.1, we obtain the following theorem.

Theorem 6.1 *Suppose that* $f(z) = h(z) + \overline{g(z)} = z + \sum_{k=n}^{\infty} a_k z^k + \sum_{k=n}^{\infty} \overline{b_k}\overline{z}^k \in \mathcal{H}$. *Then*

$$|h'(z)| + |g'(z)| \leq \frac{1 + |z|^{k-1}}{1 - |z|^{k-1}}, \;\; Reh'(z) \geq \frac{1 - |z|^{k-1}}{1 + |z|^{k-1}} + |g'(z)|,$$

and

$$\int_0^{|z|} \frac{1 - t^{k-1}}{1 + t^{k-1}} dt \leq ||h(z)| - |g(z)|| \leq |h(z)| + |g(z)| \leq \int_0^{|z|} \frac{1 + t^{k-1}}{1 - t^{k-1}} dt.$$

Corollary 6.2 *Suppose that* $f = h + \overline{g} \in \mathcal{H}$, *and* $h''(0) = g''(0) = 0$. *Then the image domain covers the disc* $|\omega| < (\pi/2) - 1$.

By using similar arguments as that of Theorem 5.1 and [7, Theorem 7], we have the following theorem.

Theorem 6.3 *Suppose that* $0 < r < 1$, $f_k(z) = \int_0^3 (1 + t^{k-1})/(1 - t^{k-1})dt$. *The area of the image of* $|z| < r$ *for mappings* $f(z) = h(z) + \overline{g(z)} = z + \sum_{k=n}^{\infty} a_k z^k + \sum_{k=n}^{\infty} \overline{b_k}\overline{z}^k$ *in* $\mathcal{H}$ *is maximal for* f_k.

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