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On Some Recurrence Relations of Generalized q-Mittag Leffler Function

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Abstract

In this paper, we investigate the q-difference relation of q- analogue of generalized Mittag Leffler function by using technique of q- calculus and also investigate some properties by using q- derivative.

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1. Introduction

In 1903, the Swedish mathematician Gosta Mittag Leffler [5] introduced the function $E_\alpha(z)$ by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad (\alpha \in C\ , \mathrm{Re}(\alpha) > 0) \tag{1.1}$$

The generalization of $E_\alpha(z)$ was studied by Wiman [14], who defined the function $E_{\alpha,\beta}(z)$ as below

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \qquad (\alpha,\beta,\gamma \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0) \tag{1.2}$$

In 1971, Prabhakar [6] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$, $\alpha,\beta,\gamma \in C$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$ which is defined by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} . \frac{z^n}{n!} \tag{1.3}$$

where $(\lambda)_n$ is the Pochhammer symbol [7] defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & n=0, \lambda \neq 0 \\ \lambda(\lambda+1)\ ...\ (\lambda+n-1), & n \in N, \lambda \in \mathbb{C} \end{cases} \tag{1.4}$$

Where N being the set of positive integers.

Another generalization of Mittag Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ was studied by T.O. Salim [9], who define the function $E^{\gamma,\delta}_{\alpha,\beta}(z)$ as follows:

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} . \frac{z^n}{(\delta)_n} \tag{1.5}$$

We state below the q-analogue of above discussed generalized Mittag – Leffter function $E^{\gamma,\delta}_{\alpha,\beta}(z;q)$ as follows

Definition 1 : For α,β,γ∈C, Re(α) > 0, Re(β) > 0, Re(γ) > 0 and |q| < 1 the function $E^{\gamma,\delta}_{\alpha,\beta}(z;q)$ is defined as

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(q^{\gamma};q)_n}{(q^{\delta};q)_n} . \frac{z^n}{\Gamma_q(\alpha n+\beta)} \tag{1.6}$$

where $\Gamma_q(\lambda)$ is the q-gamma function.

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined by

$$(\lambda;q)_n = \prod_{k=0}^{n-1}(1-\lambda q^k) = \frac{(\lambda;q)_\infty}{(\lambda q^n;q)_\infty} \tag{1.7}$$

and the q-analogue of the power $(a-b)^n$ is

$$(a-b)^0 = 1,\ (a-b)^n = \prod_{k=0}^{n-1}(a-bq^k) \tag{1.8}$$

There is following relationship between them :

$$(a-b)^n = a^n \left({}^{b}\!/_{a};q \right)_n,\ (a \neq 0)$$

$$= a^n \frac{\left({}^{b}\!/_{a};q \right)_\infty}{\left(q^n\, {}^{b}\!/_{a};q \right)_\infty} \tag{1.9}$$

Also, Predrag M. Rajkovic, et. al. [8], define a q-derivative of a function f(z) by

$$D_q f(z) = \frac{f(z)-f(qz)}{z-qz} \quad (z \neq 0) \tag{1.10}$$

Further, the $\Gamma_q(z)$ satisfies the functional equation,

$$\Gamma_q(z+1)=\frac{1-q^z}{1-q}\Gamma_q(z) \tag{1.11}$$

The detailed account of generalized Mittag-Leffler function can be found in research monographs due to Agrawal [1], Kilbas, et. al. [3], Gupta and Debnath [2], Shukla and Prajapati [11, 12, 13].and Sharma and Jain[10].

In this paper, the motive is to evaluate the recurrence relation and the recurrencerelation with q-derivative.

2. Recurrence Relations

Theorem 1 : If α,β,γ∈C, Re(α) > 0, Re(β) > 0, Re(γ) > 0 then

$$E_{\alpha,\beta}^{\gamma,\delta}(z;q) = E_{\alpha,\beta}^{\gamma+1,\delta}(z;q)-\frac{q^{\gamma}}{(1-q^{\delta})}zE_{\alpha,\alpha+\beta}^{\gamma+1,\delta+1}(z;q) - \frac{q^{\gamma+1}}{(1-q^{\delta})}zE_{\alpha,\alpha+\beta}^{\gamma+1,\delta+1}(qz;q) \tag{2.1}$$

Proof : From (1.6), we write

$$E_{\alpha,\beta}^{\gamma,\delta}(z;q)=\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_n}{(q^{\delta};q)_n}.\frac{z^n}{\Gamma_q(\alpha n+\beta)} = \frac{1}{\Gamma_q(\beta)}+\sum_{n=1}^{\infty}\frac{(q^{\gamma};q)_n}{(q^{\delta};q)_n}.\frac{z^n}{\Gamma_q(\alpha n+\beta)}$$

$$=\frac{1}{\Gamma_q(\beta)}+\sum_{n=1}^{\infty}\frac{(1-q^{\gamma})(q^{\gamma+1};q)_{n-1}}{(q^{\delta};q)_n}.\frac{z^n}{\Gamma_q(\alpha n+\beta)}$$

Since $(1-q^{\gamma})=(1-q^{\gamma+n})-q^{\gamma}(1-q^n)$ then, the above equation becomes equal to

$$\frac{1}{\Gamma_q(\beta)}+\sum_{n=1}^{\infty}\frac{(q^{\gamma+1};q)_{n-1}}{(q^{\delta};q)_n}.\frac{[(1-q^{\gamma+n})-q^{\gamma}(1-q^n)]z^n}{\Gamma_q(\alpha n+\beta)}$$

$$=\frac{1}{\Gamma_q(\beta)}+\sum_{n=1}^{\infty}\frac{(q^{\gamma+1};q)_n}{(q^{\delta};q)_n}.\frac{z^n}{\Gamma_q(\alpha n+\beta)}-\frac{q^{\gamma}}{(1-q^{\delta})}\sum_{n=1}^{\infty}\frac{(q^{\gamma+1};q)_{n-1}}{(q^{\delta+1};q)_{n-1}}.\frac{(1-q^n)z^n}{\Gamma_q(\alpha n+\beta)}$$

$$=\sum_{n=0}^{\infty}\frac{(q^{\gamma+1};q)_n}{(q^{\delta};q)_n}.\frac{z^n}{\Gamma_q(\alpha n+\beta)}-\frac{q^{\gamma}}{(1-q^{\delta})}\sum_{n=1}^{\infty}\frac{(q^{\gamma+1};q)_{n-1}}{(q^{\delta+1};q)_{n-1}}.\frac{(1-q^n)z^n}{\Gamma_q(\alpha n+\beta)}$$

$$=\sum_{n=0}^{\infty}\frac{(q^{\gamma+1};q)_n}{(q^{\delta};q)_n}.\frac{z^n}{\Gamma_q(\alpha n+\beta)}-\frac{q^{\gamma}}{(1-q^{\delta})}\sum_{n=1}^{\infty}\frac{(q^{\gamma+1};q)_{n-1}}{(q^{\delta+1};q)_{n-1}}.\frac{z^n}{\Gamma_q(\alpha n+\beta)}$$

$$-\frac{q^{\gamma}}{(1-q^{\delta})}\sum_{n=1}^{\infty}\frac{(q^{\gamma+1};q)_{n-1}}{(q^{\delta+1};q)_{n-1}}.\frac{(qz)^n}{\Gamma_q(\alpha n+\beta)}$$

On replacing n by m+1 in second and third summation, the RHS of above equation becomes

$$\sum_{n=0}^{\infty}\frac{(q^{\gamma+1};q)_n}{(q^{\delta};q)_n}.\frac{z^n}{\Gamma_q(\alpha n+\beta)}-\frac{q^{\gamma}}{(1-q^{\delta})}z\sum_{m=0}^{\infty}\frac{(q^{\gamma+1};q)_m}{(q^{\delta+1};q)_m}.\frac{z^m}{\Gamma_q[\alpha m+(\alpha+\beta)]}$$
$$-\frac{q^{\gamma+1}}{(1-q^{\delta})}z\sum_{m=0}^{\infty}\frac{(q^{\gamma+1};q)_m}{(q^{\delta+1};q)_m}.\frac{(qz)^m}{\Gamma_q[\alpha m+(\alpha+\beta)]}$$

In view of definition (1.6), the above expression becomes

$$E_{\alpha,\beta}^{\gamma+1,\delta}(z;q)-\frac{q^{\gamma}}{(1-q^{\delta})}zE_{\alpha,\alpha+\beta}^{\gamma+1,\delta+1}(z;q)-\frac{q^{\gamma+1}}{(1-q^{\delta})}zE_{\alpha,\alpha+\beta}^{\gamma+1,\delta+1}(qz;q)$$

This completes the proof of the result (2.1).

Theorem 2 : Let α,β,γ,ω∈C, then for any n = 1, 2, 3, ...

$$D_q^n[z^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta}(\omega z^{\alpha};q)]=z^{\beta-n-1}E_{\alpha,\beta-n}^{\gamma,\delta}(\omega z^{\alpha};q) \qquad (2.2)$$

where Re(β) > n.

Proof : Consider the function

$f(z)=z^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta}(\omega z^{\alpha};q)$ in (1.10) and applying the definition (1.6)

$D_q[z^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta}(\omega z^{\alpha};q)]$ becomes

$$\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_n}{(q^{\delta};q)_n}\frac{(1-q^{\alpha n+\beta-1})}{(1-q)}\frac{\omega^n z^{\alpha n+\beta-2}}{\Gamma_q(\alpha n+\beta)}$$

According to the functional equation (1.11) the above expression becomes

$$\sum_{n=0}^{\infty}\frac{(q^{\gamma};q)_n}{(q^{\delta};q)_n}\frac{\omega^n z^{\alpha n+\beta-2}}{\Gamma_q(\alpha n+\beta-1)}$$

which equals $z^{\beta-2}E_{\alpha,\beta-1}^{\gamma,\delta}(\omega z^{\alpha};q)$

Finally, we obtain

$$D_q[z^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta}(\omega z^{\alpha};q)]=z^{\beta-2}E_{\alpha,\beta-1}^{\gamma,\delta}(\omega z^{\alpha};q)$$

Iterating this result, upto n–1 times, we obtain the required formula.

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