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Structures of n-Lie algebra A^n

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Abstract

In this paper, we discuss the structure of the exterior direct sum n-Lie algebra $(A^n, [,\cdots,,]_k)$ of an n-Lie algebra A. And it is proved that, (1) if $I_1, \cdots, I_{n-1}$ are ideals of an n-Lie algebra A, then the vector space $(I_1, I_2, \cdots, I_{k-1}, I_1, I_{k+1}, \cdots, I_{n-1})$ is also an ideal of $(A^n, [,\cdots,,]_k)$, and if I is a solvable (nilpotent) ideal of A, then I^n is also solvable (nilpotent). (2) For a linear mapping $\delta \in End(A)$, then δ is a derivation of A if and only if $f_\delta \in Hom(A, A^n)$ is an n-Lie algebra homomorphism. (3) If (V, ρ) is an A-module, then $(V^n, \bar{\rho})$ is an A^n-module.

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1 Preliminary

In the paper [1], authors provided the exterior direct sum n-Lie algebras of n-Lie algebras [2, 3]. In this paper, we mainly study the structures of the exterior direct sum n-Lie algebra of a given n-Lie algebra. First, we recall some notions. Let A be a vector space. The direct sum vector space of A is $A^n = \{(x_1, \cdots, x_n) \mid x_i \in A, 1 \leq i \leq n\}$, satisfying that for all $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n) \in A^n$ and $\lambda \in F$,

$$X + Y = (x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n),$$

$$\lambda X = \lambda(x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n).$$

An n-Lie algebra [3] is a vector space A over a field F endowed with an n-ary multilinear skew-symmetric multiplication satisfying that for all $x_1, \cdots, x_n$, $y_2, \cdots, y_{n-1} \in A$,

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n} [x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]. \quad (1)$$

The identity (1) is usually called the n-Jacobi identity.

Let A be an n-Lie algebra. A derivation of an n-Lie algebra A is a linear mapping $D : A \to A$ satisfying that

$D([x_1, \cdots, x_n]) = \sum_{i=1}^{n} [x_1, \cdots, D(x_i), \cdots, x_n]$, $\forall x_1, \cdots, x_n \in A$.

By Eq.(1), for $x_1, \cdots, x_{n-1} \in A$, the left multiplication $ad(x_1, \cdots, x_{n-1}) : A \to A$ defined by for all $x \in A$, $ad(x_1, \cdots, x_{n-1})(x) = [x_1, \cdots, x_{n-1}, x]$ is a derivation of A. All the derivations of A, denoted by $Der(A)$, is a subalgebra of the general linear algebra $gl(A)$.

Let A be an n-Lie algebra and V be a vector space. If there exists a linear mapping $\rho : A^{\wedge(n-1)} \to End(V)$ satisfying that for all $x_i, y_i \in A, i = 1, \cdots, n$,

$$\rho([x_1, \cdots, x_n], y_2, \cdots, y_{n-1})$$
$$= \sum_{i=1}^{n} (-1)^{n-i} \rho(x_1, \cdots, \hat{x}_i, \cdots, x_n) \rho(x_i, y_2, \cdots, y_{n-1}), \quad (2)$$

$$[\rho(x_1, \cdots, x_{n-1}), \rho(y_1, \cdots, y_{n-1})] = \sum_{i=1}^{n} \rho(y_1, \cdots, [x_1, \cdots, x_{n-1}, y_i], \cdots, y_{n-1}) \quad (3)$$

then (V, ρ) is called a representation of A, or V is an A-module [4].

As an example, the linear mapping $\rho : A^{\wedge 2} \to End(A)$ defined by for all $x_1, \cdots, x_{n-1} \in A$, $\rho(x_1, \cdots, x_{n-1}) = \mathrm{ad}(x_1, \cdots, x_{n-1})$, (A, ad) is an A-module, which is called the *adjoint module* of A.

Let A be an n-Lie algebra and V be a subspace of A. If V satisfies that $[V, \cdots, V] \subseteq V$, then V is a subalgebra of the n-Lie algebra A. If V satisfies that $[V, A, \cdots, A] \subseteq V$, then V is called an ideal of the n-Lie algebra A. If V satisfies that $[V, \cdots, V] = 0$ ($[V, V, A, \cdots, A] = 0$), then V is called an abelian subalgebra (an abelian ideal).

2 Structures of n-Lie algebra A^n

Lemma 2.1[1] *Let A be an n-Lie algebra. Then for any $s \geq 2$, A^n is an n-Lie algebra in the multiplication $[,\cdots,]_s$, where for all $X_j = (x_1^j,\cdots,x_n^j) \in A^n$, $j = 1,\cdots,n$,*

$$[X_1,\cdots,X_n]_s = (\sum_{i=1}^{n}[x_s^1,\cdots,x_1^i,\cdots,x_s^n],[x_2^1,\cdots,x_2^n],\cdots,[x_n^1,\cdots,x_n^n]). \quad (4)$$

The n-Lie algebra $(A^n,[,\cdots,]_s)$ is called the exterior direct sum n-Lie algebra. For the similarity, in the following, we mainly discuss the case $s = 2$.

Theorem 2.2 *Let A be an n-Lie algebra, I_i, $i = 1,\cdots,n-1$ be ideals of A. Then*

$$U = (I_1, I_1, I_2,\cdots,I_{n-1}) = \{(x_1,\cdots,x_n) \mid x_1,x_2 \in I_1,\ x_i \in I_i, 3 \leq i \leq n\}$$

is an ideal of $(A^n,[,\cdots,]_2)$.

Proof For all $(y_1,\cdots,y_n) \in U$, $x_i^j \in A, 1 \leq i \leq n$, $2 \leq j \leq n$, by Eq.(4),

$$[(y_1,\cdots,y_n),(x_1^2,\cdots,x_n^2),\cdots,(x_1^n,\cdots,x_n^n)]_2$$
$$= ([y_1,x_2^2,\cdots,x_2^n],[y_2,x_2^2,\cdots,x_2^n],\cdots,[y_n,x_n^2,\cdots,x_n^n])$$
$$+(\sum_{l=2}^{n}[y_2,\cdots,\underbrace{x_1^l}_{l},\cdots,x_2^n],[y_2,x_2^2,\cdots,x_2^n],\cdots,[y_n,x_n^2,\cdots,x_n^n]).$$

Since I_j for $j = 1,\cdots,n-1$ are ideals of A and $y_1,y_2 \in I_1$, we obtain that $[(y_1,\cdots,y_n),(x_1^2,\cdots,x_n^2),\cdots,(x_1^n,\cdots,x_n^n)]_2 \in U$. It follows the result.

Theorem 2.3 *Let A be an n-Lie algebra, $I_1,\cdots,I_{n-1}$ be ideals of the n-Lie algebra A. Then for any $3 \leq k \leq n$, $U_k = (I_1, I_2,\cdots,I_{k-1}, I_1, I_{k+1},\cdots,I_{n-1})$ is an ideal of the n-Lie algebra $(A_n,[,\cdots,]_k)$.*

Proof The proof is similar to Theorem 2.2.

Theorem 2.4 *Let A be an n-Lie algebra, I be a solvable (nilpotent) ideal of A. Then I^n is a solvable (nilpotent) ideal of the n-Lie algebras $(A^n,[,\cdots,]_k)$, $2 \leq k \leq n$. Especially, if I is an abelian ideal of the n-Lie algebra A, then $W = (I,\cdots,I)$ is an abelian ideal.*

Proof By Theorem 2.2, I^n is an ideal of n-Lie algebras $(A^n,[,\cdots,]_k)$, $3 \leq k \leq n$. We only need to prove the solvability and the nilpotency. Since the similarity, we only prove the case $k = 2$. Denote $W = (I,\cdots,I)$. By hypothesis, there exists a number $s \geq 0$ such that $I^{(s)} = 0$. We will show that for any $r \geq 0$, $W^{(r)} \subseteq (I^{(r)},\cdots,I^{(r)})$.

For all $y_i^l \in I$, $x_i^j \in A$, $1 \leq l \leq 2; 3 \leq j \leq n; 1 \leq i \leq n$, suppose

$$[(y_1^1,\cdots,y_n^1),(y_1^2,\cdots,y_n^2),(x_1^3,\cdots,x_n^3),\cdots,(x_1^n,\cdots,x_n^n)]_2 = (z_1,\cdots,z_n).$$

Thanks to Eq.(4), for $2 \leq t \leq n$, $z_t = [y_t^1,y_t^2,x_t^3,\cdots,x_t^n] \in I^{(1)}$, and

$$z_1 = [y_1^1,y_2^2,x_2^3,\cdots,x_2^n] + [y_2^1,y_1^2,x_2^3,\cdots,x_2^n] + \sum_{v=3}^{n}[y_2^1,y_2^2,\cdots,x_1^v,\cdots,x_2^n] \in I^{(1)}.$$

Therefore, $z_t \in I^{(1)}$ for $1 \le t \le n$, we obtain $W^{(1)} \subseteq (I^{(1)}, \cdots, I^{(1)})$.

Now suppose $W^{(s-1)} \subseteq (I^{(s-1)}, \cdots, I^{(s-1)})$. By Theorem 2.2 and a similar discussion, we have

$W^{(s)} = [W^{(s-1)}, W^{(s-1)}, A^n, \cdots, A^n]_2 \subseteq (I^{(s)}, \cdots, I^{(s)})$.

Since $I^{(s)} = 0$, we have $W^{(s)} = 0$, that is, W is solvable.

Similar discussion, if I is nilpotent, then W is a nilpotent ideal of the exterior direct sum n-Lie algebra A^n.

If I is an abelian ideal, then $[I, I, A, \cdots, A] = 0$. Then for all $X_i = (x_1^i, \cdots, x_n^i) \in A^n$, $1 \le i \le n$, where $X_1, X_2 \in I^n$, by Eq.(4), $[X_1, X_2, X_3, \cdots, X_n]_2 = 0$. Therefore, I^n is an abelian ideal. The proof is complete.

Now we discuss the relation between derivations of A with $(A^n, [,\cdots,]_k)$, for $k \ge 3$. Since the similarity of the discussion, we only study the case $k = 2$.

For convenience, in the following the exterior direct sum n-Lie algebra $(A^n, [,\cdots,]_2)$ of an n-Lie algebra A is simply denoted by A_n.

Theorem 2.5 *Let A be an n-Lie algebra, $\delta \in End(A)$. Define linear mapping $f_\delta : A \to A^n$ by the formula*

$$f_\delta(x) = (\delta x, x, \cdots, x), \forall x \in A. \tag{5}$$

Then δ is a derivation of A if and only if f_δ is an algebra homomorphism.

Proof If δ is a derivation of A. Then for all $x_i \in A$, $i = 1, \cdots, n$, by Eq.(4) and Eq.(5),

$f_\delta([x_1, x_2, \cdots, x_n]) = (\delta([x_1, x_2, \cdots, x_n]), [x_1, \cdots, x_n], \cdots, [x_1, \cdots, x_n])$,

$[f_\delta(x_1), \cdots, f_\delta(x_n)]_2$

$= [(\delta(x_1), \cdots, x_1), (\delta(x_2), \cdots, x_2), \cdots, (\delta(x_n), \cdots, x_n)]_2$

$= (\sum\limits_{l=1}^{n} [x_1, \cdots, \delta(x_i), \cdots, x_n], [x_1, \cdots, x_n], \cdots, [x_1, \cdots, x_n])$.

Since $\delta([x_1, x_2, \cdots, x_n]) = \sum\limits_{i=1}^{n} [x_1, \cdots, \delta(x_i), \cdots, x_n]$, we have

$f_\delta([x_1, x_2, \cdots, x_n]) = [f_\delta(x_1), \cdots, f_\delta(x_n)]_2$.

Conversely, if f_δ is an n-Lie algebra homomorphism, then for all $x_i \in A$, $1 \le i \le n$, $[f_\delta(x_1), \cdots, f_\delta(x_n)] = f_\delta([x_1, \cdots, x_n])$. Thanks to Eq.(4) and Eq.(5),

$[(\delta(x_1), \cdots, x_1), \cdots, (\delta(x_n), \cdots, x_n)]_2$

$= (\sum\limits_{l=1}^{n} [x_1, \cdots, \delta(x_i), \cdots, x_n], [x_1, x_2, \cdots, x_n], \cdots, [x_1, x_2, \cdots, x_n])$

$= (\delta([x_1, x_2, \cdots, x_n]), \cdots, [x_1, x_2, \cdots, x_n])$.

Therefore, $\delta([x_1, x_2, \cdots, x_n]) = \sum\limits_{l=1}^{n} [x_1, \cdots, \delta(x_i), \cdots, x_n]$, that is, δ is a derivation of A. The proof is complete.

At last of the paper, we study the representation of the exterior direct sum n-Lie algebras. Let A be an n-Lie algebra, V be a vector space and $\rho : A^{\wedge n-1} \to End(V)$ be a linear mapping. By the paper [4], (V, ρ) is an n-Lie algebra A-module if and only if the direct sum vector space $B = A \oplus V$ is an

n-Lie algebra in the following multiplication, for all $x_i \in A, v \in V$, $1 \le i \le n$,

$$[x_1, \cdots, x_n]_B = [x_1, \cdots, x_n], \quad [x_1, \cdots, x_{n-1}, v]_B = \rho(x_1, \cdots, x_{n-1})v,$$

and V is an abelian ideal, that is, $[A, \cdots, A, V, V]_B = 0$. Then we have the following result.

Theorem 2.6 *Let A be an n-Lie algebra, (V, ρ) be a representation of n-Lie algebra A. Then $(V^n, \bar{\rho})$ is a representation of the exterior direct sum n-Lie algebra A^n, where the linear mapping $\bar{\rho} : (A^n)^{\wedge n-1} \to End(V^n)$ defined by for all $X_i = (x_1^i, \cdots, x_n^i) \in A^n$, $1 \le i \le n-1$, and $u = (u_1, \cdots, u_n) \in V^n$,*

$$\bar{\rho}(X_1, \cdots, X_{n-1})u = (\sum_{i=1}^{n-1} \rho(x_2^1, \cdots, x_1^i, \cdots, x_2^{n-1})u_2$$

$$+\rho(x_2^1, \cdots, x_2^{n-1})u_1, \rho(x_2^1, \cdots, x_2^{n-1})u_2, \cdots, \rho(x_n^1, \cdots, x_n^{n-1})u_n).$$

Proof Since (V, ρ) is a representation of A, then $(B = A \oplus V, [, \cdots,]_B)$ is an n-Lie algebra. Therefore, we obtain the exterior direct sum n-Lie algebra $(B^n, [, \cdots,]_2)$ of the n-Lie algebra $(B = A \oplus V, \quad [, \cdots,]_B)$. From V is an abelian ideal of B, and Theorem 2.2, V^n is an abelian ideal of the n-Lie algebra $(B^n, [, \cdots,]_2)$.

Define linear mapping $\bar{\rho} : (A^n)^{\wedge n-1} \to End(V^n)$ by for all $X_1, \cdots, X_{n-1} \in A^n$, $w = (w_1, \cdots, w_n) \in V^n$,

$$\bar{\rho}(X_1, \cdots, X_{n-1})(w) = ad_{B^n}(X_1, \cdots, X_{n-1})(w) = [X_1, \cdots, X_{n-1}, w]_2.$$

By a direct computation, $\bar{\rho}$ satisfies Eq.(2) and Eq.(3). Therefore, $(V^n, \bar{\rho})$ is a representation of A^n. The proof is complete.

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