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On the Generalized Ceiling and Floor Distributions

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Abstract

This paper deals with three new generalized ceiling density functions, their distribution functions and moments. Graphical representations of the density and distribution functions are also given for various values of the parameters. The distributions functions, for the two corresponding generalized floor density functions given earlier, are also derived and represented graphically for various values of the parameters. Expression for moments are also given.

Mathematics Subject Classification: 60E05, 62B15, 33C60, 60E10.

Keywords: Generalized ceiling distribution, generalized floor distribution, moments.

1 Introduction

In this paper, three new generalized ceiling distributions are introduced and their distribution functions and moments are derived. The density and distribution functions are graphically shown for various values of the parameters.

The distribution functions and moments for the corresponding two generalized floor distributions given earlier in [1] are also derived. The distribution functions are graphically represented for various values of the parameters.

2 Generalized Ceiling Distributions

This section deals with three generalized ceiling distributions. A few properties and graphical representations of the density and distribution functions are also given.

2.1 Generalized Ceiling Distribution - I

The generalized ceiling distribution $g_1(x)$ of a random variable X, for $x \geq 1$, is defined by

$$g_1(x) = d_1(a,b)x^{-a}e^{b\lceil lnx\rceil}, \tag{1}$$

with $x \geq 1, a > 1+b$, where

$$d_1(a,b) = \frac{(a-1)\left(1-e^{-(a-1-b)}\right)}{e^b\left(1-e^{-(a-1)}\right)}. \tag{2}$$

The constant $d_1(a,b)$ is determined from

$$\begin{aligned} 1 &= d_1(a,b)\int_1^\infty x^{-a}e^{b\lceil lnx\rceil}dx \\ &= d_1(a,b)\int_0^\infty e^{-(a-1)y+b\lceil y\rceil}dy, \end{aligned}$$

by substituting $lnx = y$. Thus,

$$\begin{aligned} 1 &= d_1(a,b)\sum_{n=0}^{\infty}e^{b(n+1)}\int_n^{n+1}e^{-(a-1)y}dy \\ &= \frac{d_1(a,b)}{1-a}\sum_{n=0}^{\infty}e^{b(n+1)}\left[e^{-(a-1)(n+1)}-e^{-(a-1)n}\right] \\ &= \frac{d_1(a,b)e^b\left(e^{-(a-1)}-1\right)}{(1-a)\left(1-e^{-(a-1-b)}\right)}. \end{aligned} \tag{3}$$

Hence (3) implies (2). The r-th moments about the origin is given by

$$E(X^r) = \frac{d_1(a,b)}{d_1(a-r,b)}, \quad r < a-1-b. \tag{4}$$

The expression for mean, variance, kurtosis and skewness may be obtained from (4). The distribution function $G_1(x)$ is given by

$$\begin{aligned} G_1(x) &= d_1(a,b)\int_1^x x^{-a}e^{b\lceil lnx\rceil}dx \\ &= d_1(a,b)\int_0^{lnx}e^{-(a-1)y+b\lceil y\rceil}dy \\ &= d_1(a,b)\left[\sum_{n=0}^{\lceil lnx\rceil-2}e^{b(n+1)}\int_n^{n+1}e^{-(a-1)y}dy + e^{b\lceil lnx\rceil}\int_{\lceil lnx\rceil-1}^{lnx}e^{-(a-1)y}dy\right], \end{aligned}$$

thus,

$$\begin{aligned} G_1(x) &= \frac{d_1(a,b)}{1-a}\left(\frac{e^b\left(e^{-(a-1)}-1\right)\left(1-e^{-(a-1-b)(\lceil lnx\rceil-1)}\right)}{1-e^{-(a-1-b)}}\right) \\ &+ \frac{d_1(a,b)e^{b\lceil lnx\rceil}}{1-a}\left(e^{-(a-1)lnx}-e^{-(a-1)(\lceil lnx\rceil-1)}\right) \quad (5) \end{aligned}$$

for $x \geq 1$, $a > 1+b$, where $d_1(a,b)$ is given by (2). The density $g_1(x)$ and the distribution function $G_1(x)$, for some values of the parameters a and b, are plotted in Figure 1.

2.2 Generalized Ceiling Distribution - II

We define the generalized ceiling distribution $g_2(x)$ of a random variable X, for $x > 0$, as

$$g_2(x) = d_2(a,b)x^{-a}e^{-b\lceil |lnx|\rceil}, \quad x > 0. \quad (6)$$

The constant $d_2(a,b)$ is obtained from

$$\begin{aligned} \frac{1}{d_2(a,b)} &= \int_0^\infty x^{-a}e^{b\lceil |lnx|\rceil}dx \\ &= \int_{-\infty}^\infty e^{-(a-1)y+b\lceil |y|\rceil}dy \\ &= \int_0^\infty e^{-(a-1)y-b\lceil y\rceil}dy + \int_0^\infty e^{(a-1)y-b\lceil y\rceil}dy \\ &= \frac{1}{d_1(a,-b)} + \frac{1}{d_1(2-a,-b)}, \end{aligned}$$

where $d_1(a,b)$ is given in (2). Thus,

$$d_2(a,b) = \frac{d_1(a,-b)d_1(2-a,-b)}{d_1(a,-b)+d_1(2-a,-b)}, \quad (7)$$

for $1-b < a < 1+b$. The r-th moments about the origin is given by

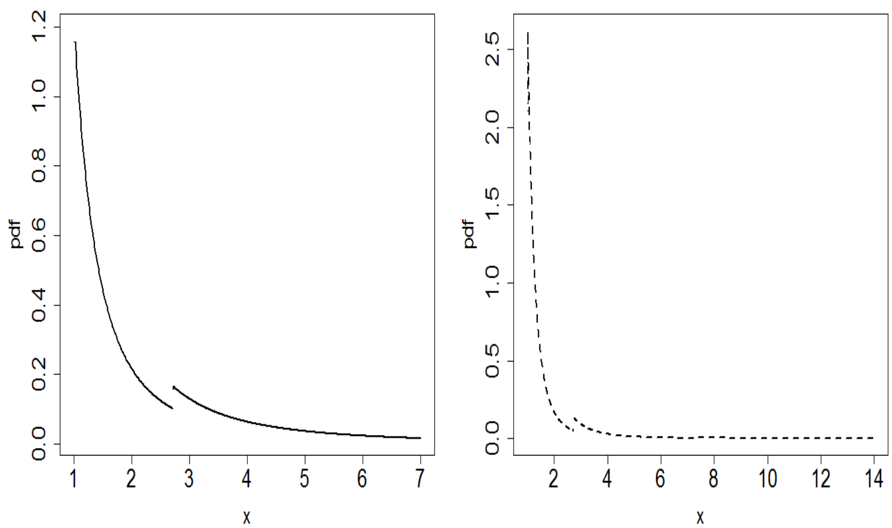
$$E(X^r) = \frac{d_2(a,b)}{d_2(a-r,b)}, \quad (8)$$

for $a-1-b < r < a-1+b$.

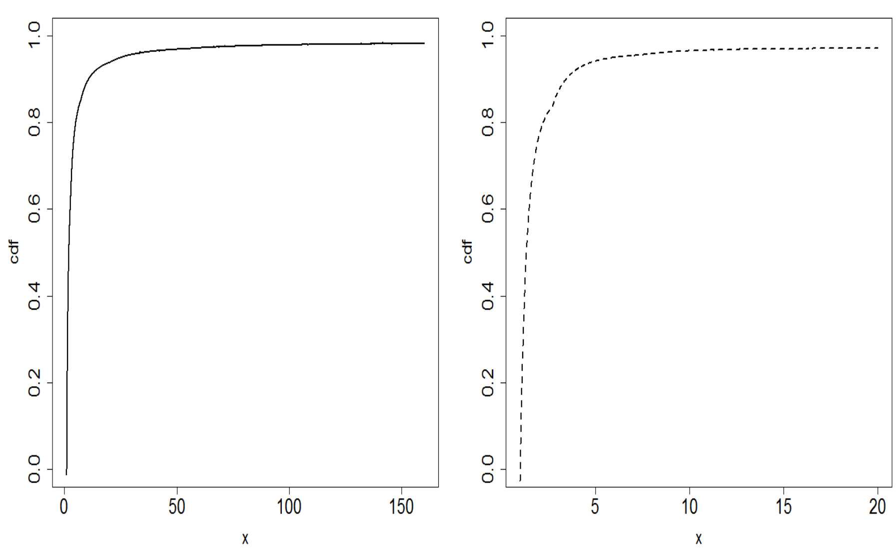
The corresponding distribution function $G_2(x)$ is obtained from

$$\begin{aligned} G_2(x) &= d_2(a,b)\int_0^x x^{-a}e^{-b\lceil |lnx|\rceil}dx \\ &= d_2(a,b)\int_{-\infty}^{lnx} e^{-(a-1)y}e^{-b\lceil |y|\rceil}dy. \end{aligned}$$

Thus, $G_2(x)$ is obtained separately for the following two cases:



(a) Left: $a = 2.5$ and $b = 0.5$. Right: $a = 4$ and $b = 1$



(b) Left: $a = 2.5$ and $b = 0.5$. Right: $a = 4$ and $b = 1$

Figure 1: Plots of $g_1(x)$ and $G_1(x)$ for some values of the parameters a and b.

- Subcase 1: $x > 1$;
- Subcase 2: $0 < x < 1$.

Subcase 1: $x > 1$

For $x > 1$, $G_2(x)$ is given by

$$\begin{aligned} G_2(x) &= d_2(a,b)\left[\int_{-\infty}^{0} e^{-(a-1)y}e^{-b\lceil |y|\rceil}dy + \int_{0}^{lnx} e^{-(a-1)y}e^{-b\lceil y\rceil}dy\right] \\ &= d_2(a,b)\left[\int_{0}^{\infty} e^{(a-1)y}e^{-b\lceil y\rceil}dy + \int_{0}^{lnx} e^{-(a-1)y}e^{-b\lceil y\rceil}dy\right] \\ &= d_2(a,b)[I_1+I_2], \qquad (9) \end{aligned}$$

suppose. Now,

$$\begin{aligned} I_1 &= \sum_{n=0}^{\infty} e^{-b(n+1)} \int_{n}^{n+1} e^{(a-1)y}dy \\ &= \frac{e^{-b}\left(e^{a-1}-1\right)}{(a-1)\left(1-e^{-(b-a+1)}\right)}, \quad b > a-1, \qquad (10) \end{aligned}$$

as done in earlier subsection, and

$$\begin{aligned} I_2 &= \sum_{n=0}^{\lceil lnx\rceil-2} e^{-b(n+1)} \int_{n}^{n+1} e^{(a-1)y}dy + e^{-b\lceil lnx\rceil} \int_{\lceil lnx\rceil-1}^{lnx} e^{-(a-1)y}dy \\ &= \frac{e^{-b}\left(e^{-(a-1)}-1\right)\left(1-e^{-(a-1+b)(\lceil lnx\rceil-1)}\right)}{(1-a)\left(1-e^{-(a-1+b)}\right)} \\ &+ \frac{e^{-b\lceil lnx\rceil}}{1-a}\left[x^{1-a}-e^{-(a-1)(\lceil lnx\rceil-1)}\right], \qquad (11) \end{aligned}$$

as done in earlier subsection.

Subcase 2: $0 < x < 1$

In this subcase,

$$G_2(x) = d_2(a,b)\int_{-lnx}^{\infty} e^{(a-1)y}e^{-b\lceil |y|\rceil}dy = d_2(a,b)I_3,$$

suppose. Here,

$$\begin{aligned}
I_3 &= \int_{\lceil -lnx \rceil}^{\infty} e^{(a-1)y} e^{-b\lceil y \rceil} dy + \int_{-lnx}^{\lceil -lnx \rceil} e^{(a-1)y} e^{-b\lceil y \rceil} dy \\
&= \sum_{n=\lceil -lnx \rceil}^{\infty} e^{-b(n+1)} \int_{n}^{n+1} e^{(a-1)y} dy + e^{-b\lceil -lnx \rceil} \int_{-lnx}^{\lceil -lnx \rceil} e^{(a-1)y} dy \\
&= \sum_{n=\lceil -lnx \rceil}^{\infty} e^{-b(n+1)} \left[\frac{e^{(a-1)(n+1)} - e^{(a-1)n}}{a-1} \right] + e^{-b\lceil -lnx \rceil} \left[\frac{e^{(a-1)\lceil -lnx \rceil} - e^{(a-1)(-lnx)}}{a-1} \right] \\
&= \frac{1}{a-1} \left[e^{-b} \left(e^{a-1} - 1 \right) \sum_{n=\lceil -lnx \rceil}^{\infty} e^{-(b-a+1)n} + e^{-b\lceil -lnx \rceil} \left(e^{(a-1)\lceil -lnx \rceil} - e^{(a-1)(-lnx)} \right) \right] \\
&= \frac{e^{-b} \left(e^{a-1} - 1 \right)}{a-1} \left(\frac{1}{1 - e^{-(b-a+1)}} - \frac{1 - e^{-(b-a+1)\lceil -lnx \rceil}}{1 - e^{-(b-a+1)}} \right) \\
&+ \frac{e^{-b\lceil -lnx \rceil}}{a-1} \left(e^{(a-1)\lceil -lnx \rceil} - e^{(a-1)(-lnx)} \right), \qquad (12)
\end{aligned}$$

for $b > a-1$. For some values of the parameters a and b, $g_2(x)$ and $G_2(x)$ are shown in Figure 2.

2.3 Generalized Ceiling Distribution - III

The generalized ceiling density function $g_3(x)$ of a random variable X, for $-\infty < x < \infty$, is given by

$$g_3(x) = d_3(a,b)|x|^{-a} e^{-b\lceil |ln|x|| \rceil}, \qquad (13)$$

for $-\infty < x < \infty$, $1-b < a < 1+b$. This is a symmetric density about the origin. The constant $d_3(a,b)$ is obtained below. We have

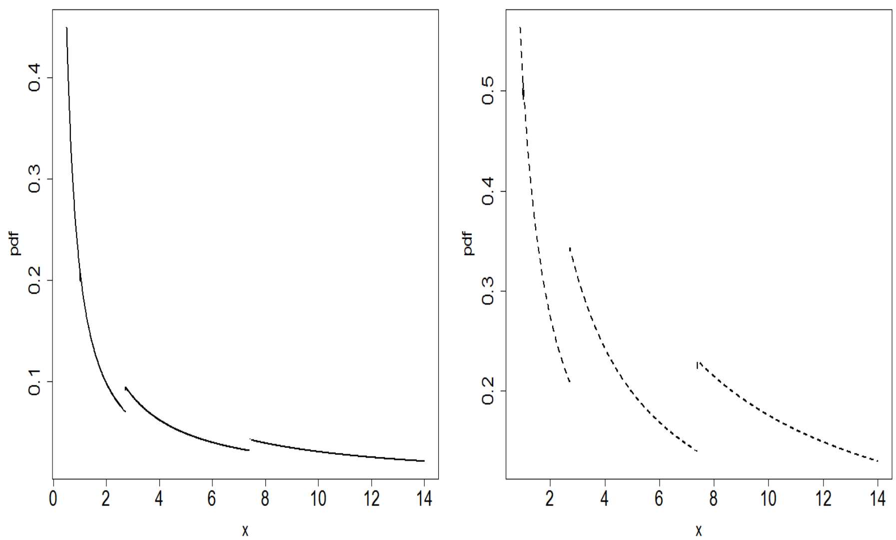
$$\begin{aligned}
\frac{1}{d_3(a,b)} &= \int_{-\infty}^{\infty} |x|^{-a} e^{-b\lceil |ln|x|| \rceil} dx \\
&= 2 \int_{0}^{\infty} x^{-a} e^{-b\lceil |lnx| \rceil} dx \\
&= \frac{2}{d_2(a,b)},
\end{aligned}$$

using (6). Thus,

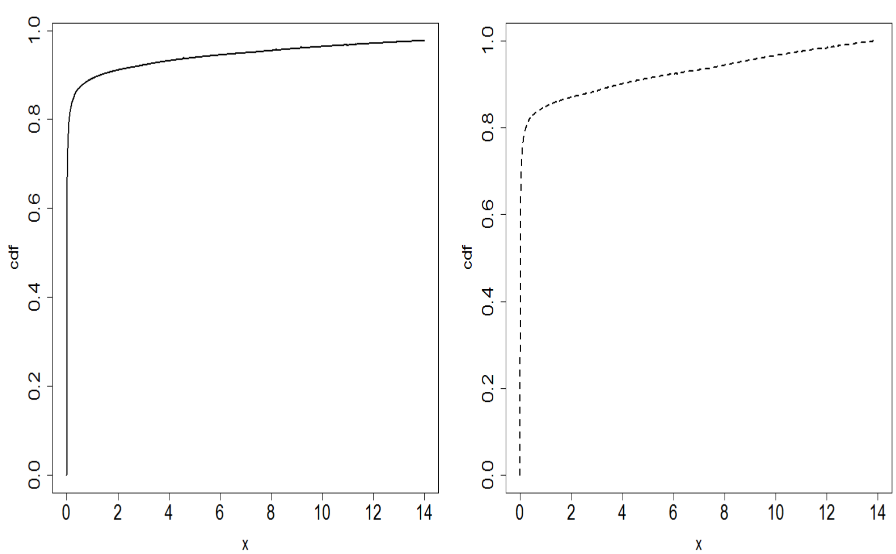
$$d_3(a,b) = \frac{d_2(a,b)}{2}. \qquad (14)$$

It is easy to see that

$$E(X^r) = \begin{cases} \frac{d_2(a,b)}{d_2(a-r,b)}, & \text{for } r \text{ even integer, and } a-1-b < r < a-1+b \\ 0, & \text{for } r \text{ odd integer.} \end{cases} \qquad (15)$$



(a) Left: $a = 1.1$ and $b = 0.3$. Right: $a = 0.9$ and $b = 0.5$



(b) Left: $a = 1.1$ and $b = 0.3$. Right: $a = 0.9$ and $b = 0.5$

Figure 2: Plots of $g_2(x)$ and $G_2(x)$ for some values of the parameters a and b.

The distribution function $G_3(x)$ is given by

$$\begin{aligned} G_3(x) &= d_3(a,b)\int_{-\infty}^{x} |x|^{-a}e^{-b\lceil|ln|x||\rceil}dx \\ &= \begin{cases} \frac{1}{2}+d_3(a,b)I_x, & x>0, \\ \frac{1}{2}-d_3(a,b)I_x, & x<0. \end{cases} \end{aligned} \tag{16}$$

where

$$\begin{aligned} I_x &= \int_0^x y^{-a}e^{-b\lceil|lny|\rceil}dy \\ &= \begin{cases} I_1+I_2, & x>1, \\ I_3, & 0<x<1. \end{cases} \end{aligned} \tag{17}$$

with I_1, I_2 and I_3 given respectively in (10), (11) and (12). The density and distribution functions are shown in Figure 3 for some values of the parameters.

3 Generalized Floor Distributions

In this section, the three generalized floor distributions given earlier in [1] are mentioned. The moments about origin and the distribution functions are obtained for the second and third generalized floor distributions. These results for the first generalized floor distribution are already available in [1].

3.1 Generalized Floor Distribution - I

The density function of the first generalized floor distribution $f_1(x)$, as given in [1], is

$$f_1(x) = C_1(a,b)x^{-a}e^{-b\lfloor lnx\rfloor}, \tag{18}$$

for $x \geq 1, a > b+1$, where

$$C_1(a,b) = \frac{(a-1)\left(e^{b+1}-e^a\right)}{e-e^a}. \tag{19}$$

For some values of the parameters a and b, $f_1(x)$ and $F_1(x)$ are shown in Figure 4.

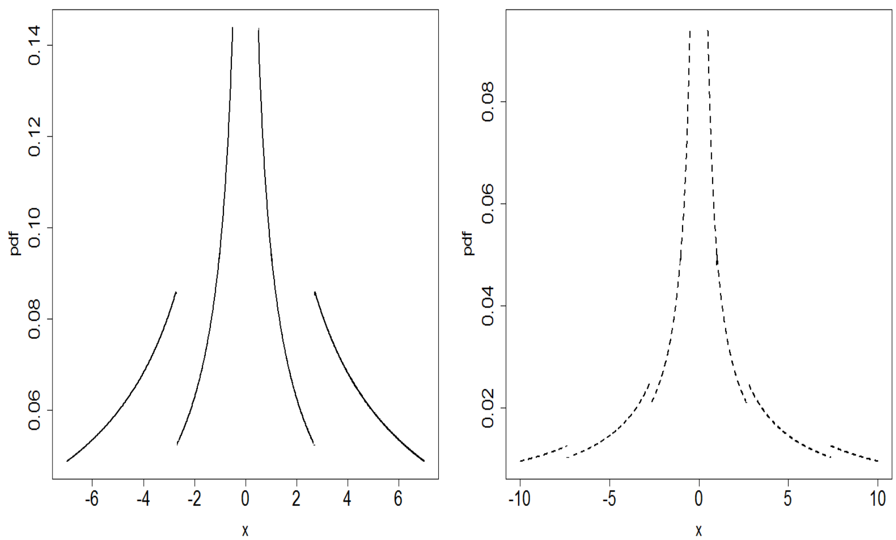
3.2 Generalized Floor Distribution - II

The second generalized floor distribution, as defined in [1], has the density function given by

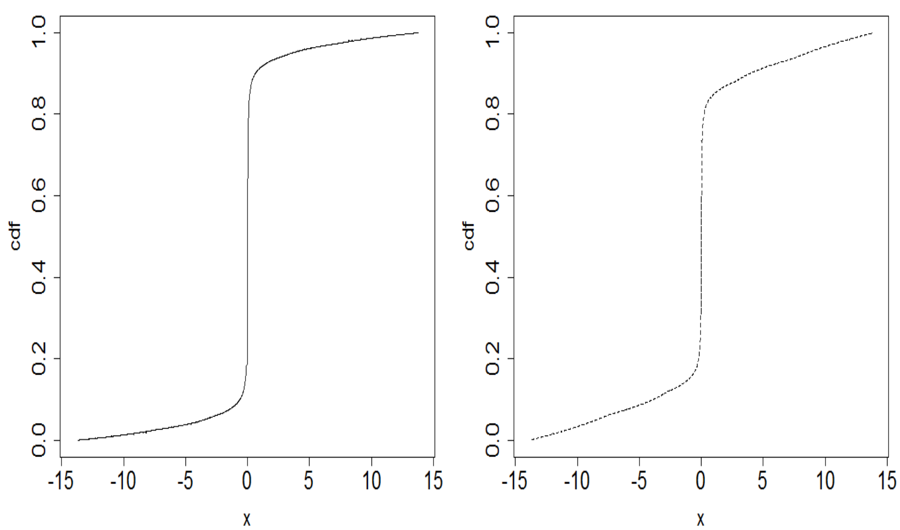
$$f_2(x) = C_2(a,b)x^{-a}e^{-b\lfloor|lnx|\rfloor}, \tag{20}$$

for $a \geq 0, b > 0, 1-b < a < 1+b, 0 < x < \infty$, where

$$C_2(a,b) = \frac{(a-1)\left(1-e^{a-b-1}\right)\left(e^{a+b-1}-1\right)}{\left(e^{2(a-1)}-1\right)\left(e^b-1\right)}. \tag{21}$$

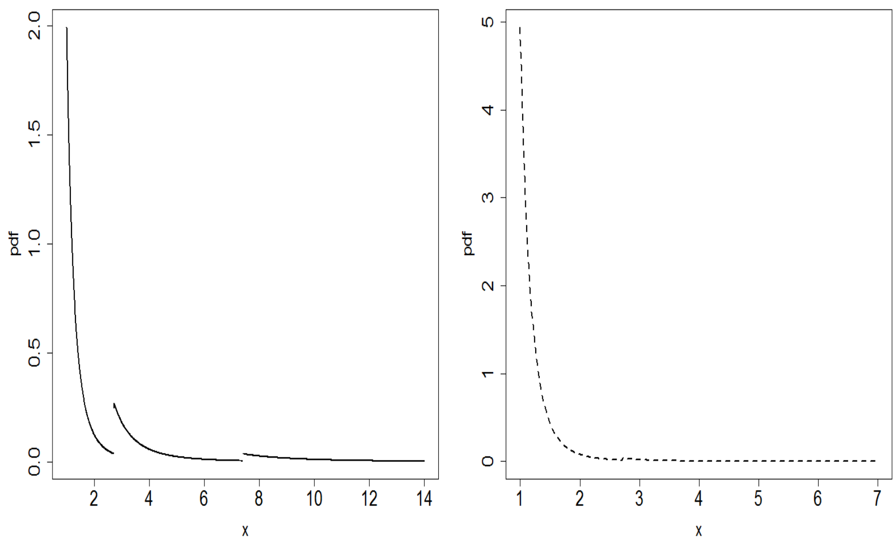


(a) Left: $a = 0.6$ and $b = 0.5$. Right: $a = 0.9$ and $b = 0.2$

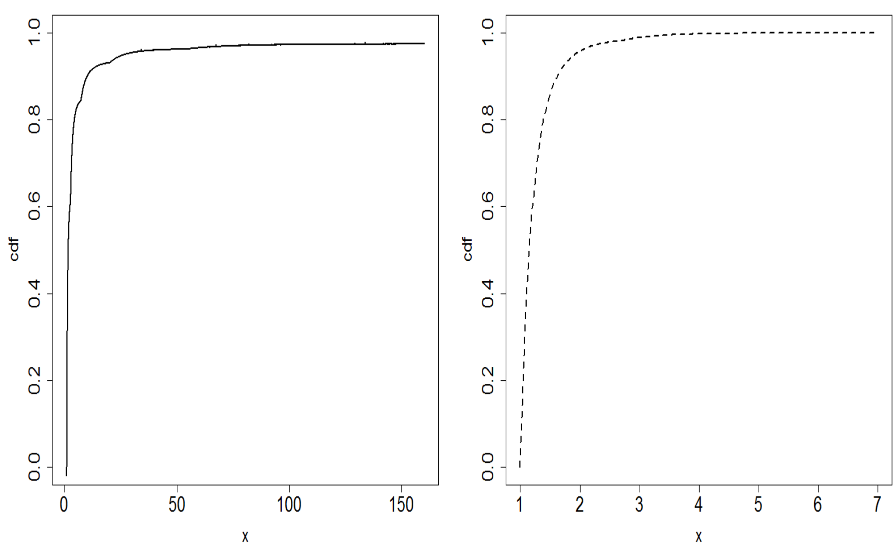


(b) Left: $a = 0.6$ and $b = 0.5$. Right: $a = 0.9$ and $b = 0.2$

Figure 3: Plots of $g_3(x)$ and $G_3(x)$ for some values of the parameters a and b.



(a) Left: $a = 4$ and $b = 2$. Right: $a = 6$ and $b = 1$



(b) Left: $a = 4$ and $b = 2$. Right: $a = 6$ and $b = 1$

Figure 4: Plots of $f_1(x)$ and $F_1(x)$ for some values of the parameters a and b.

The r-th moments about the origin can be easily obtained as

$$E(X^r) = \frac{C_2(a,b)}{C_2(a-r,b)}, \tag{22}$$

for $a-1-b<r<a-1+b$.

The distribution function is given by

$$\begin{aligned} F_2(x) &= C_2(a,b)\int_0^x x^{-a}e^{-b\lfloor|lnx|\rfloor}dx \\ &= C_2(a,b)\int_{-\infty}^{lnx} e^{-(a-1)y}e^{-b\lfloor|y|\rfloor}dy. \end{aligned}$$

Following the procedure adopted for the generalized ceiling distribution $G_2(x)$, we obtain $F_2(x)$, for $x>1$, as

$$\begin{aligned} F_2(x) &= \frac{C_2(a,b)}{a-1}\frac{\left(e^{-(1-a)}-1\right)}{\left(1-e^{-(b+1-a)}\right)} - \frac{C_2(a,b)}{a-1}\frac{\left(e^{-(a-1)}-1\right)\left(1-e^{-(b+a-1)\lfloor lnx\rfloor}\right)}{\left(1-e^{-(b+a-1)}\right)} \\ &- \frac{C_2(a,b)e^{-b\lfloor lnx\rfloor}}{a-1}\left(x^{1-a}-e^{-(a-1)\lfloor lnx\rfloor}\right), \end{aligned} \tag{23}$$

for $x>1$ and $1-b<a<1+b$. Also

$$\begin{aligned} F_2(x) &= \frac{C_2(a,b)}{a-1}\frac{\left(e^{a-1}-1\right)\left(1-e^{-(b-a+1)\lfloor -lnx\rfloor}\right)}{1-e^{-(b-a+1)}} \\ &- \frac{C_2(a,b)e^{-b\lfloor -lnx\rfloor}}{a-1}\left(x^{1-a}-e^{(a-1)\lfloor -lnx\rfloor}\right) \end{aligned} \tag{24}$$

for $0<x<1$, and $b-a+1>0$. $f_2(x)$ and $F_2(x)$ are plotted in Figure 5 for some values of the parameters a and b for $0<x<\infty$.

3.3 Generalized Floor Distribution - III

The third generalized floor symmetric distribution defined in [1] has the following form

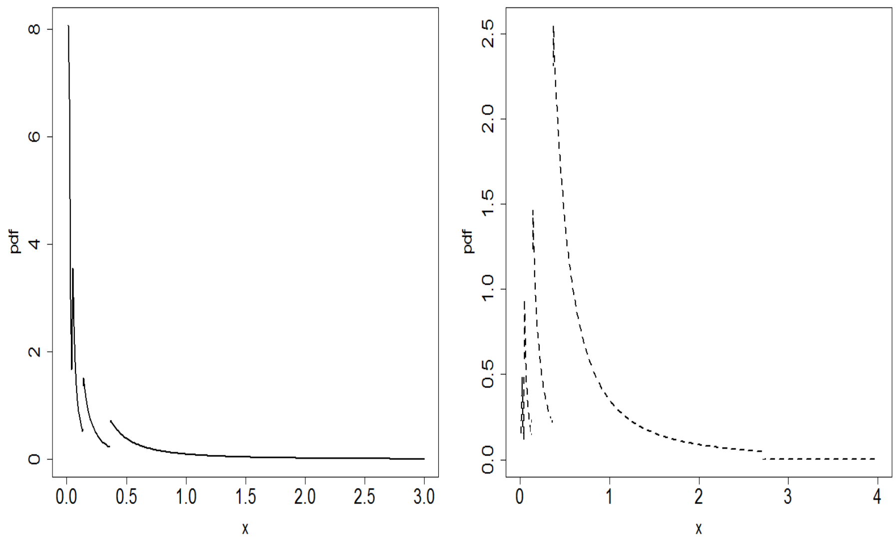
$$f_3(x) = C_3(a,b)|x|^{-a}e^{-b\lfloor|lnx|\rfloor}, \tag{25}$$

for $a\geq 0, b>0, 1-b<a<1+b, -\infty<x<\infty$, where

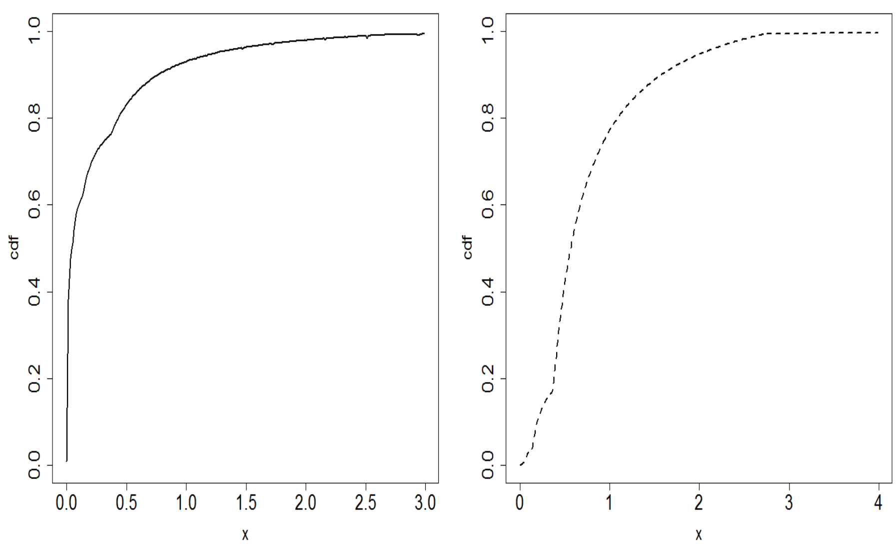
$$C_3(a,b) = \frac{(a-1)\left(1-e^{a-b-1}\right)\left(e^{a+b-1}-1\right)}{2\left(e^{2(a-1)}-1\right)\left(e^b-1\right)} = \frac{C_2(a,b)}{2}. \tag{26}$$

It is easy to obtain the r-th absolute moments as

$$E(|X|^r) = \frac{C_3(a,b)}{C_3(a-r,b)} = \frac{C_2(a,b)}{C_2(a-r,b)}, \tag{27}$$



(a) Left: $a = 2$ and $b = 1.2$. Right: $a = 2$ and $b = 2.5$



(b) Left: $a = 2$ and $b = 1.2$. Right: $a = 2$ and $b = 2.5$

Figure 5: Plots of $f_2(x)$ and $F_2(x)$ for some values of the parameters a and b.

for $a-1-b<r<a-1+b$.

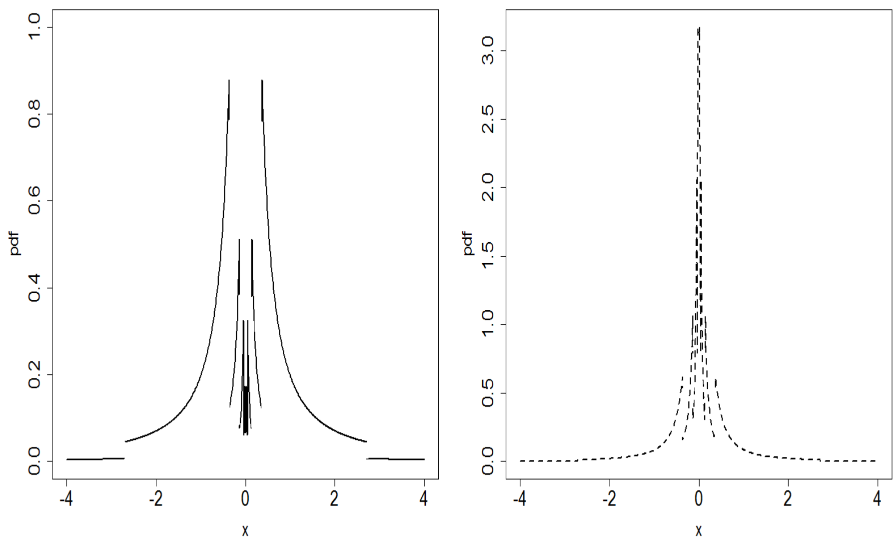
The corresponding distribution function, $F_3(x)$, is given by

$$\begin{aligned} F_3(x) &= C_3(a,b)\int_{-\infty}^{x}|x|^{-a}e^{-b\lfloor|lnx|\rfloor}dx \\ &= \begin{cases}\frac{1}{2}+C_3(a,b)I_x, & x>0,\\ \frac{1}{2}-C_3(a,b)I_x, & x<0.\end{cases} \end{aligned} \tag{28}$$

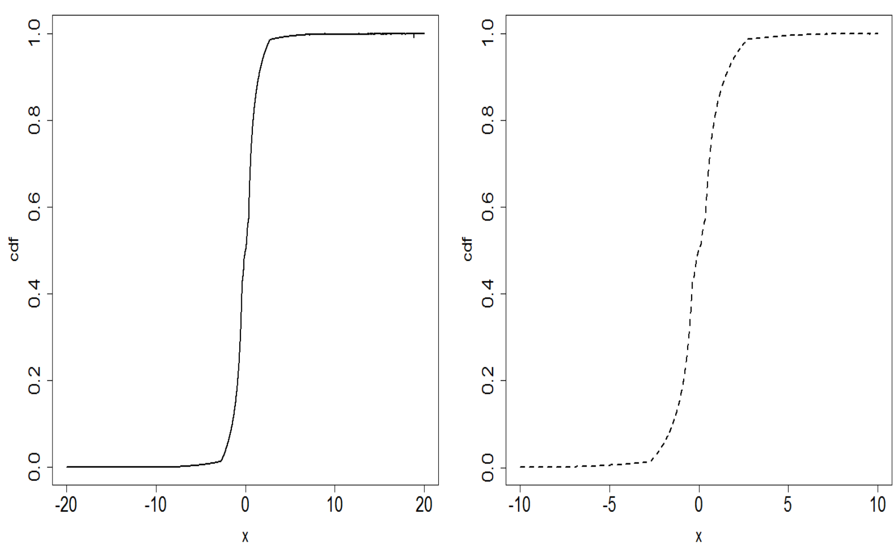
where

$$\begin{aligned} I_x &= \int_{0}^{x}y^{-a}e^{-b\lfloor|lny|\rfloor}dy \\ &= \int_{-\infty}^{lnx}e^{-(a-1)z}e^{-b\lfloor|z|\rfloor}dz. \end{aligned} \tag{29}$$

The values of I_x for $x>1$ is obtained from the right hand side expression of (23) by omitting $C_2(a,b)$ and for $0<x<1$ from (24) by omitting $C_2(a,b)$. The graphs for $f_3(x)$ and $F_3(x)$ for some values of the parameters are given in Figure 6.



(a) Left: $a = 1.5$ and $b = 2$. Right: $a = 2$ and $b = 1.4$



(b) Left: $a = 1.5$ and $b = 2$. Right: $a = 2$ and $b = 1.4$

Figure 6: Plots of $f_3(x)$ and $F_3(x)$ for some values of the parameters a and b.

4 Concluding Remarks

The ceiling and the floor distributions may be heavy-tailed distributions. The study of these distributions from "Regular Variation", Bingham *et al.* [2], point of view along with their possible statistical inference analysis and applications will be discussed in a future paper.

Further possible generalizations, their properties and applications will also be subject matter for future research.

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[2] N.H. Bingham, C.M. Goldie and J.L. Tangels. Regular Variation, *Vol. 27 of Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, 1987.

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