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Regularity of Weak Solutions to Some Anisotropic Elliptic Equations

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Abstract

We consider boundary value problem of the form This paper we deals with the system of N partial differential equations

$$\sum_{i=1}^{n} D_i(a_i^\alpha(x, Du(x))) = \sum_{i=1}^{n} D_i F_i^\alpha(x), \quad \alpha = 1, \cdots N.$$

We show that regularity of boundary datum u_* forces u to have regularity as well, provided we assume suitable ellipticity and growth conditions on a_i^α.

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1 Introduction

In a recent paper [1], Leonetti and Petricca considered the system of N partial differential equations

$$\sum_{i=1}^{n} D_i(a_i^\alpha(x, Du(x))) = 0, \ x \in \Omega, \ \alpha = 1, 2, \cdots, N. \tag{1.1}$$

Under the boundary condition

$$u(x) = u_*(x), \quad x \in \partial\Omega, \tag{1.2}$$

the authors showed that higher integrability of the boundary datum u_* forces solutions u to have higher integrability as well, provided a_i^α satisfy suitable ellipticity and growth conditions. Among all the results, they obtained a theorem see [1, Theorem 1.3].

In this paper, we consider a more general problem. We refer the reader to [1] for the notations and symbols used in this paper. Let us consider the following system of N partial differential equations

$$\sum_{i=1}^n D_i(a_i^\alpha(x, Du(x))) = \sum_{i=1}^n D_i F_i^\alpha(x), \quad \alpha = 1, \cdots, N, \tag{1.3}$$

and suppose that the Carathéodory functions $a_i^\alpha(x,z) : \Omega \times \mathrm{R}^n \to \mathrm{R}$ satisfy the monotonicity condition

$$\nu \sum_{i=1}^n |z_i - \tilde{z}_i|^{p_i} \le \sum_{i=1}^n (a_i^\alpha(x,z) - a_i^\alpha(x,\tilde{z}))(z_i^\alpha - \tilde{z}_i{}^\alpha), \tag{1.4}$$

for some positive constant ν,and

$$|a_i^\alpha(x,z)| \le c(1 + \sum_{j=1}^n |z_j|^{p_j})^{1-\frac{1}{p_i}}, i = 1, 2, \cdots, n, \quad \alpha = 1, 2, \cdots, N, \tag{1.5}$$

We work in Marcinkiewicz spaces: if $q > 1$, then the space $M^m(\Omega)$ consists of measurable functions g on Ω such that

$$\sup_{t>0} t|\{x \in \Omega : |g(x)| > t\}|^{\frac{1}{m}} < \infty.$$

This condition is equivalently stated as

$$|||g(x)|||_m = \sup_{E \subset \Omega, |E|>0} \frac{1}{|E|^{\frac{1}{m'}}} \int_E |g(x)| dx < \infty,$$

where m' is the conjugate exponent of q, $\frac{1}{m} + \frac{1}{m'} = 1$. We recall that $L^q(\Omega)$ is a proper subspace of $L^q_{weak}(\Omega)$, and if $g \in L^q_{weak}(\Omega)$ for some $q > 1$, then $g \in L^{q-\varepsilon}(\Omega)$ for every $0 < \varepsilon \le q - 1$.

The following Sobolev type inequality is also proved: there exists a positive constant c, depending only on Ω, such that

$$\|v\|_{L^r(\Omega)} \le c \prod_{i=1}^n \|\partial_i v\|^{\frac{1}{n}}_{L^{p_i}(\Omega)}, \qquad \forall r \in [1, \overline{p}^*], \tag{1.6}$$

for any v$\in C_0^1(\Omega)$ where p_i >1 for $i = 1, 2, \cdots, n$. In the following the letter c will freely denote a constant, not necessarily the same in any two occurrences, while only the relevant dependence will be highlighted. In this paper we present some results concerning the case of f belonging to a Marcinkiewicz space, M^m

of (1.1), with $m > p_{\infty}^{'}$, we consider the case $p_\infty = \bar{p}^*$, where $p_\infty = \max\{p_n, \bar{p}^*\}$, $p_n = \max\{p_i\}$, and we also consider the elliptic systems problem with $f \in M^m$ too, not only this but also our method is different from the previous one.For some recent developments on anisotropic functionals and anisotropic elliptic equations and systems, see[2-3].

2 Preliminary Notes

We mow introduce some symbols used in this paper. Let $T_k(u)$ is the usual truncation of u at level k >0, that is,

$$T_k(u) = \max\{-k, \min\{k, u\}\}.$$

Moreover, let

$$G_k(u) = u - T_k(u).$$

Definition 2.1 *A function* $u \in u_* + W_0^{1,(p_i)}(\Omega; R^N)$ *is called a solution to the boundary value problem*

$$\begin{cases} \sum\limits_{i=1}^{n} D_i(a_i^\alpha(x, Du(x))) = f^\alpha(x), & x \in \Omega, \\ u(x) = u_*(x), & x \in \partial\Omega, \end{cases} \tag{2.7}$$

if

$$\int_\Omega \sum_{i=1}^{n} \sum_{\alpha=1}^{N} a_i^\alpha(x, Du(x)) \cdot D_i\varphi^\alpha(x)dx = \int_\Omega \sum_{\alpha=1}^{N} f^\alpha(x) \cdot \varphi^\alpha(x)dx, \tag{2.8}$$

holds true for any $\varphi \in W_0^{1,(p_i)}(\Omega; R^N), \alpha = 1, 2, \cdots, N$.

3 Main Results

These are the main results of the paper.

Theorem 3.1 *Let* $f \in M^m(\Omega; R^N)$, $u_* \in W^{1,1}(\Omega; R^N)$ *with* $D_i u_* \in M^{p_i m}$, $i = 1, 2, \cdots, n$, *and under previous assumptions* $(1.4) - (1.5)$,
i) If m> $\frac{n}{\overline{p}}$, *then there exists a weak solution u for the problem (1.3), and* $u - u_*$ *is bounded;*
ii) If m= $\frac{n}{\overline{p}}$, *then there exists a weak solution u for the problem (1.3) and a constant* $\beta > 0$ *such that*

$$\int_\Omega e^{\beta|u-u_*|} < \infty;$$

iii) If $(\overline{p}^*)' < m < \frac{n}{\overline{p}}$, *then there exists a weak solution u for the problem*(1.3) *and* $u - u_*$ *belongs to* M^s *with s satisfies*

$$s = \frac{m\overline{p}^*(\overline{p} - 1)}{m\overline{p} + \overline{p}^* - m\overline{p}^*} = \frac{mn(\overline{p} - 1)}{n - m\overline{p}}. \tag{3.9}$$

Proof of Theorem 3.1. Let us fix the component $\beta\in\{1,2,\cdots,N\}$ and let us consider the test function

$$\varphi=(0,\cdots,0,\varphi^{\beta},0,\cdots,0),$$

where, for $k\in(0,+\infty)$ we take $\varphi^{\beta}=G_k(u^{\beta}-u_*^{\beta})$ in (2.8). We have

$$\sum_{i=1}^{n}\int_{\Omega}a_i^{\beta}(x,Du)D_iG_k(u^{\beta}-u_*^{\beta})=\int_{\Omega}f^{\beta}G_k(u^{\beta}-u_*^{\beta}),$$

because

$$\sum_{i=1}^{n}\int_{\Omega}a_i^{\beta}(x,Du)D_iG_k(u^{\beta}-u_*^{\beta})=\sum_{i=1}^{n}\int_{A_k}a_i^{\beta}(x,Du)D_i(u^{\beta}-u_*^{\beta})=\int_{A_k}f^{\beta}(u^{\beta}-u_*^{\beta}),$$

where $A_k=\{|u^{\beta}-u_*^{\beta}|>k\}$. Hence by (1.4), (1.5) and Young inequality we obtain that

$$\begin{aligned}
&\tilde{\nu}\sum_{i=1}^{n}\int_{A_k}|D_iu^{\beta}-D_iu_*^{\beta}|^{p_i}\\
\le\ &\sum_{i=1}^{n}\int_{A_k}(a_i^{\beta}(x,Du)-a_i^{\beta}(x,Du_*))(D_iu^{\beta}-D_iu_*^{\beta})\\
=\ &\int_{A_k}f^{\beta}(u^{\beta}-u_*^{\beta})-\sum_{i=1}^{n}\int_{A_k}a_i^{\beta}(x,Du_*^{\beta})(D_iu^{\beta}-D_iu_*^{\beta})\\
\le\ &\int_{A_k}|f^{\beta}(u^{\beta}-u_*^{\beta})|+\sum_{i=1}^{n}\int_{A_k}|a_i^{\beta}(x,Du_*)||D_iu^{\beta}-D_iu_*^{\beta}|\\
\le\ &\int_{A_k}|f^{\beta}(u^{\beta}-u_*^{\beta})|+c\sum_{i=1}^{n}\int_{A_k}(1+\sum_{j=1}^{N}|D_iu_*|^{p_j})^{1-\frac{1}{p_i}}(D_iu^{\beta}-D_iu_*^{\beta})\\
\le\ &\int_{A_k}|f^{\beta}(u^{\beta}-u_*^{\beta})|+c(\varepsilon)\sum_{i=1}^{n}\int_{A_k}(1+\sum_{i=1}^{n}|D_iu_*|^{p_i})+\varepsilon\sum_{i=1}^{n}\int_{A_k}|D_iu^{\beta}-D_iu_*^{\beta}|^{p_i}.
\end{aligned}$$

Then

$$\int_{A_k}|D_iu^{\beta}-D_iu_*^{\beta}|^{p_i}\le c(\int_{A_k}|f^{\beta}(u^{\beta}-u_*^{\beta})|+|A_k|+\sum_{i=1}^{n}\int_{A_k}|D_iu_*|^{p_i}).\quad(3.10)$$

Therefore, by (1.6), with $r=\overline{p}^*$, Hölder inequality and (3.10), we get

$$\begin{aligned}
&c(\int_{A_k}|u^{\beta}-u_*^{\beta}|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}}\\
\le\ &\prod_{i=1}^{n}(\int_{A_k}|D_iu^{\beta}-D_iu_*^{\beta}|^{p_i})^{\frac{1}{p_in}}\\
\le\ &(\int_{A_k}|f^{\beta}(u^{\beta}-u_*^{\beta})|+|A_k|+\sum_{i=1}^{n}\int_{A_k}|D_iu_*|^{p_i})^{\frac{1}{\overline{p}}}\\
\le\ &(\int_{A_k}|f^{\beta}|(\overline{p}^*)')^{\frac{1}{(\overline{p}^*)'}}(\int_{A_k}|u^{\beta}-u_*^{\beta}|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}}+|A_k|+\sum_{i=1}^{n}\int_{A_k}|D_iu_*|^{p_i}]^{\frac{1}{\overline{p}}}.
\end{aligned}\quad(3.11)$$

Since $f \in M^m(\Omega)$ and $D_i u_* \in M^{p_i m}$, and $m \geq (\overline{p}^*)'$, we have

$$\int_{A_k} |f|^{(\overline{p}^*)'} \leq c|A_k|^{1-\frac{(\overline{p}^*)'}{m}}, \int_{A_k} |D_i u_*|^{p_i} \leq c|A_k|^{1-\frac{1}{m}}.$$

Then by applying Young inequality, $|A_k|^{\frac{1}{m}} \leq |\Omega|^{\frac{1}{m}}$ and (3.11) becomes

$$\begin{aligned}
& c(\int_{A_k} |u^\beta - u_*^\beta|^{\overline{p}^*})^{\frac{\overline{p}}{\overline{p}^*}} \\
\leq\ & (|A_k|^{1-\frac{(\overline{p}^*)'}{m}})^{\frac{1}{(\overline{p}^*)'}} (\int_{A_k} |u^\beta - u_*^\beta|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}} + |A_k| + |A_k|^{1-\frac{1}{m}} \\
\leq\ & |A_k|^{\frac{1}{(\overline{p}^*)'}-\frac{1}{m}} (\int_{A_k} |u^\beta - u_*^\beta|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}} + \frac{1}{k}|A_k|^{\frac{1}{(\overline{p}^*)'}-\frac{1}{m}} \cdot |A_k|^{\frac{1}{m}} (\int_{A_k} |u^\beta - u_*^\beta|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}} \\
& + \frac{1}{k}|A_k|^{-\frac{1}{m}} \int_{A_k} |u^\beta - u_*^\beta| \\
\leq\ & |A_k|^{\frac{1}{(\overline{p}^*)'}-\frac{1}{m}} (\int_{A_k} |u^\beta - u_*^\beta|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}} + c|A_k|^{\frac{1}{(\overline{p}^*)'}-\frac{1}{m}} (\int_{A_k} |u^\beta - u_*^\beta|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}} \\
& + c|A_k|^{-\frac{1}{m}} (\int_{A_k} |u^\beta - u_*^\beta|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}} |A_k|^{\frac{1}{(\overline{p}^*)'}} \\
\leq\ & c(\varepsilon)|A_k|^{(\frac{1}{(\overline{p}^*)'}-\frac{1}{m})(\overline{p})'} + \varepsilon(\int_{A_k} |u^\beta - u_*^\beta|^{\overline{p}^*})^{\frac{\overline{p}}{\overline{p}^*}}.
\end{aligned}$$

Hence by applying Hölder inequality with exponents $\overline{p}^*$ and $(\overline{p}^*)'$ to $\int_\Omega |G_k(u^\beta - u_*^\beta)| = \int_{A_k} |u^\beta - u_*^\beta|$ and by simplifying, we obtain

$$\int_\Omega |G_k(u^\beta - u_*^\beta)| \leq c|A_k|^{(\frac{1}{(\overline{p}^*)'}-\frac{1}{m})\frac{1}{\overline{p}-1}+1-\frac{1}{\overline{p}^*}}. \tag{3.12}$$

We define $g(k) = \int_\Omega |G_k(u^\beta - u_*^\beta)|$ and we recall that $g'(k) = -|A_k|$, for almost every k (see[4], [5]). We obtain, from(3.12), that

$$g(k)^{\frac{1}{\gamma}} \leq -cg'(k),$$

with $\gamma = (\frac{1}{(\overline{p}^*)'} - \frac{1}{m})\frac{1}{\overline{p}-1} + 1 - \frac{1}{\overline{p}^*}$. Therefore

$$1 \leq -cg'(k)g(k)^{-\frac{1}{\gamma}} = -\frac{c}{1-\frac{1}{\gamma}}(g(k)^{1-\frac{1}{\gamma}})'. \tag{3.13}$$

If we are in case i) of Theorem 3.1, we note that

$$1 - \frac{1}{\gamma} > 0.$$

Therefore, by integrating (3.13) from 0 to k, we get

$$k \leq -c[g(k)^{1-\frac{1}{\gamma}} - g(0)^{1-\frac{1}{\gamma}}],$$

i.e.

$$cg(k)^{1-\frac{1}{\gamma}} \leq -k + c\|u^\beta - u_*^\beta\|_{L^1(\Omega)}^{1-\frac{1}{\gamma}}.$$

Since $g(k)$ is a non-negative and decreasing function, from the latter inequality we deduce that there exists k_0, such that g(k_0)=0, and so $u^\beta - u_*^\beta \in L^\infty(\Omega)$. In case ii) of Theorem 3.1, since $m = \frac{n}{p}$, $\gamma = 1$, we have

$$1 \leq -c\frac{g^{'}(x)}{g(x)}.$$

By integrating from 0 to k, we have

$$\frac{k}{c} \leq \log[\frac{\|u^\beta - u_*^\beta\|_{L^1(\Omega)}}{g(k)}],$$

and since the function $t \to e^t$ increases, we obtain

$$e^{\frac{k}{c}} \leq \frac{\|u^\beta - u_*^\beta\|_{L^1(\Omega)}}{g(k)} \Rightarrow g(k)e^{\frac{k}{c}} \leq \|u^\beta - u_*^\beta\|_{L^1(\Omega)}.$$

So, recalling that

$$g(k) = \int_\Omega |G_k(u^\beta - u_*^\beta)| \geq \int_{A_{2k}} |G_k(u^\beta - u_*^\beta)| \geq k|A_{2k}|, \tag{3.14}$$

Hence, if $k \geq$1, we have

$$g(k) \geq |A_{2k}| \Rightarrow |A_{2k}|e^{\frac{k}{c}} \leq \|u^\beta - u_*^\beta\|_{L^1(\Omega)}.$$

Hence, if $k \geq$2, we get

$$|A_k|e^{\frac{k}{2c}} \leq \|u^\beta - u_*^\beta\|_{L^1(\Omega)}. \tag{3.15}$$

We prove now that the previous inequality implies that

$$\sum_{k=0}^{+\infty} e^{k\tau}|A_k| < \infty,$$

with $0 < \tau < \frac{1}{2c}$. Indeed, by (3.15),

$$\sum_{k=0}^{+\infty} e^{k\tau}|A_k| \leq (1+e)|\Omega| + \sum_{k=2}^{+\infty} \frac{\|u^\beta - u_*^\beta\|_{L^1(\Omega)}}{e^{k(\frac{1}{2c}-\tau)}} < \infty.$$

Since

$$\sum_{k=0}^{+\infty} e^{k\tau}|A_k| < +\infty \Rightarrow \int_\Omega e^{\beta|u^\beta - u_*^\beta|} < +\infty,$$

ii) is proved. To conclude, we consider case iii). In this case we have

$$1 - \frac{1}{\gamma} < 0.$$

Therefore,

$$1 \leq c(g(k)^{1-\frac{1}{\gamma}})'.$$

By integration from 0 to k, we obtain

$$k \leq c[g(k)^{1-\frac{1}{\gamma}} - g(0)^{1-\frac{1}{\gamma}}] \leq cg(k)^{1-\frac{1}{\gamma}},$$

and so

$$g(k)^{-1+\frac{1}{\gamma}} \leq \frac{c}{k} \Rightarrow g(k) \leq \frac{c}{k^{\frac{\gamma}{1-\gamma}}}.$$

Therefore, by (3.14), it holds true that

$$|A_{2k}| \leq \frac{g(k)}{k} \leq \frac{c}{k^{\frac{\gamma}{1-\gamma}}k} = \frac{c}{k^{\frac{1}{1-\gamma}}}.$$

By recalling the definition of γ, we obtain

$$\frac{1}{\gamma} = \frac{mn(\overline{p}-1)}{n - m\overline{p}} = s,$$

so that $u^\beta - u_*^\beta \in M^s(\Omega)$. This ends the proof of Theorem 1.1.

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