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Extremum principle for vector valued minimizers and the weak solutions of elliptic systems

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Abstract

In this paper we consider the minimum principle for vector valued minimizers of some functionals

$$\mathcal{F}(u;\Omega)=\int_\Omega f(x,Du(x))dx.$$

The main assumption on the density $f(x,z)$is a kind of "monotonicity" with respect to the $N\times n$ matrix z. We also consider the maximum and minimum principle for weak solutions u of some elliptic systems

$$-\sum_{i=1}^n D_i(a_i^\alpha(x,u(x))=0,\quad x\in\Omega,\quad \alpha=1,\dots,N,$$

and the main assumption on $a_i^\alpha(x,z)$ is

$$0<\sum_{j=1}^n\sum_{\alpha=1}^N a_i^\alpha(x,z)(z_i^\alpha-\tilde{z}_i^\alpha),$$

where $\tilde{z}$ is a $N\times n$ matrix with respect to z.

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1 Introduction

Let us consider vector valued mappings $u{:}\Omega \subset R^n \to R^n$; when $x \in \Omega$, it turns out that $Du(x)$ is a $n \times n$ matrix. For $i \in \{1, \ldots, n\}$ we set $M_i(Du)$ to be the vector containing all the minors $i \times i$ taken from the $n \times n$ matrix Du. Thus $M_1(Du) = Du$ and $M_n(Du) = detDu$. Let consider the variational integral

$$\mathcal{F}(u;\Omega) = \int_\Omega f(x, Du(x))dx, \tag{1.1}$$

where $f(Du) = g_1(M_1(Du)) + g_2(M_2(Du)) + \cdots + g_n(M_n(Du))$. For a suitable choice of $g_i's$, such an integral is a model functional in nonlinear elasticity. We can refer to [1] for maximum principle for minimizers of some integral function like (1.1). In this paper we select two conditions on f allowing for minimum principle: the first one ensures that for every minimizers u of (1.1) there exists another minimizer $\tilde{u}$ enjoying the minimum principle; the second one is stronger than the first one and it guarantees that *every* minimizer u of (1.1) satisfies the minimum principle. Next section contains precise statements and their proofs. In the last section we deal with the maximum and minimum principle for weak solutions of some elliptic systems

$$-\sum_{i=1}^{n} D_i(a_i^\alpha(x, u(x)) = 0, \quad x \in \Omega, \quad \alpha = 1, \ldots, N, \tag{1.2}$$

with a strict monotonicity condition which ensure the weak sub(super)-solution to (1.2) satisfies the maximum(minimum) principle. We will also give the precise statements and proofs.

2 Minimum principle for vector valued minimizers

Let Ω be a bounded open subset of R^n, and $u : \Omega \subset R^n \to R^N; n, N \geq 2$. We consider the functional

$$\mathcal{F}(u;\Omega) = \int_\Omega f(x, Du(x))dx, \tag{2.1}$$

where $f : \Omega \times R^{N\times n} \to R$ is assumed to be measurable with respect to $x \in \Omega$ and continuous with respect to $z \in R^{N\times n}$.We also require that

$$0 \leq f(x, z) \tag{2.2}$$

for every $x \in \Omega$, for each $z \in R^{N\times n}$. When dealing with a matrix $z \in R^{N\times n}$, we write $z^1, \ldots, z^N$ to denote the N rows; for each row z^α it result that $z^\alpha = (z_1^\alpha, \ldots, z_n^\alpha) \in R^n$. Now we write the main assumption :

$$f(x, \tilde{z}) \leq f(x, z) \tag{2.3}$$

for every $x \in \Omega$, for every couple of matrices $\tilde{z}, z \in R^{N \times n}$, such that there exists $\beta \in \{1, \dots, N\}$ for which $\tilde{z}^\beta = 0 \neq z^\beta$ and $\tilde{z}^\alpha = z^\alpha$ for $\alpha \neq \beta$. We can refer to [2] for "monotonicity".

A minimizers of functional (2.1) is a function $u \in W^{1,1}(\Omega, R^N)$ such that $\mathcal{F}(u) < +\infty$ and

$$\mathcal{F}(u) \leq \mathcal{F}(v), \tag{2.4}$$

for every $v \in u + W_0^{1,1}(\Omega, R^N)$.
For $\gamma \in \{1, \dots, N\}$ and $b \in R$ we define the truncation operator

$$T^{\gamma,b} : R^N \to R^N \tag{2.5}$$

as follows. For every $y = (y^1, \dots, y^N) \in R^N$ we set $T^{\gamma,b}(y) = (y^1, \dots, y^\gamma \vee b, \dots, y^N)$.
The main result in this section is the following.

Theorem 2.1 *Let $u = (u^1, \dots, u^N) \in W^{1,1}(\Omega, R^N)$ be a minimizers of functional (2.1) under (2.2) and (2.3). If there exist $\beta \in \{1, \dots, N\}$ and $k \in R$ such that $u^\beta \geq k$ on $\partial\Omega$, then $T^{\beta,k}(u) \in u + W_0^{1,1}(\Omega, R^N)$ and $T^{\beta,k}(u)$ minimizers (2.1) too.*

In order to get the equality $T^{\beta,k}(u) = u$ we assume the "strict monotonicity":

$$f(x, \tilde{z}) < f(x, z), \tag{2.6}$$

for every $x \in \Omega$, for every couple of matrices $\tilde{z}, z \in R^{N \times n}$, such that there exists $\beta \in \{1, \dots, N\}$ for which $\tilde{z}^\beta = 0 \neq z^\beta$ and $\tilde{z}^\alpha = z^\alpha$ for $\alpha \neq \beta$. Under (2.6) we are able to prove that $T^{\beta,k}(u) = u$ in Theorem 2.1, that is, *every* minimizers enjoys the minimum principle: that is the second result of this section.

Theorem 2.2 *Let $u = (u^1, \dots, u^N) \in W^{1,1}(\Omega, R^N)$ be a minimizers of functional (2.1) under (2.2) and (2.6). If there exist $\beta \in \{1, \dots, N\}$ and $k \in R$ such that*

$$u^\beta \geq k \quad on \quad \partial\Omega,$$

then

$$u^\beta \geq k \quad in \quad \Omega.$$

We can refer to [1-3] for more details.
Now we prove the Theorems.
Proof of Theorem 2.1. Let $\beta \in \{1, \dots, N\}$ and $k \in R$ such that $u^\beta \geq k$ on $\partial\Omega$. Set

$$\varphi^\beta = -min\{u^\beta - k, 0\}.$$

Since $u^\beta \geq k$ on $\partial\Omega$, it turns out that $\varphi^\beta \in W_0^{1,1}(\Omega)$. If $\alpha \neq \beta$ we simply set $\varphi^\alpha = 0$. Then we have $\varphi \in W_0^{1,1}(\Omega, R^N)$ and

$$\tilde{u} := u + \varphi \in W_0^{1,1}(\Omega, R^N) \tag{2.7}$$

is a test function for the minimality condition (2.4). Set

$$\Omega_1 = \{x \in \Omega : u^\beta \geq k\} \cup \{x \in \Omega : u^\beta < k, \quad Du^\beta(x) = 0\}$$

and

$$\Omega_2 = \Omega \setminus \Omega_1.$$

Then

$$D\tilde{u} = Du \quad on \quad \Omega_1 \tag{2.8}$$

and

$$D\tilde{u}^\alpha = \begin{cases} Du^\alpha, & if \quad \alpha \neq \beta, \\ 0 \neq Du^\beta, & if \quad \alpha = \beta \end{cases} \quad on \quad \Omega_2. \tag{2.9}$$

Thus

$$f(x, D\tilde{u}(x)) = f(x, Du(x)) \quad if \quad x \in \Omega_1 \tag{2.10}$$

and, using "monotonicity" (2.3),

$$f(x, D\tilde{u}(x)) \leq f(x, Du(x)) \quad if \quad x \in \Omega_2. \tag{2.11}$$

The previous (2.10), (2.11) and the positivity (2.2) merge into

$$0 \leq \mathcal{F}(\tilde{u}) \leq \mathcal{F}(u). \tag{2.12}$$

Since $\mathcal{F}(u) < +\infty$, it turns out that $\mathcal{F}(\tilde{u}) < +\infty$ too. Moreover, the minimality (2.4) of u gives

$$\mathcal{F}(u) \leq \mathcal{F}(\tilde{u})$$

thus

$$\mathcal{F}(\tilde{u}) = \mathcal{F}(u) = \min_{v \in u + W_0^{1,1}(\Omega, R^N)} \mathcal{F}(v) \tag{2.13}$$

and $\tilde{u}$ turns out to be a a minimizer too. Note that

$$\tilde{u} = T^{\beta,k}(u).$$

This ends of the proof of Theorem 2.1.
Proof of Theorem 2.2. We argue as in the proof of Theorem 2.1 until we reach (2.13). Because of (2.8), the equality (2.13) reads as

$$\int_{\Omega_2} f(x, Du(x)) = \int_{\Omega_2} f(x, D\tilde{u}(x)). \tag{2.14}$$

Now the "strict monotonicity" (2.6) can be used with $\tilde{z} = D\tilde{u}(x)$ and $z = Du(x)$,because of (2.10):

$$f(x, D\tilde{u}(x)) < f(x, Du(x)) \quad if \quad x \in \Omega_2. \tag{2.15}$$

Comparing (2.13) with (2.14) gives that Ω_2 has zero measure. This means that $Du^\beta(x) = 0$ for almost every $x \in \{u^\beta < k\}$, thus $D\varphi^\beta(x) = 0$ a.e. in Ω. Since $\varphi^\beta \in W_0^{1,1}(\Omega)$, by Poincaré inequality it follows that $\varphi^\beta = 0$ for a.e. $x \in \Omega$. Since $\varphi^\beta = -min\{u^\beta - k, 0\} > 0$ on $\{u^\beta < k\}$, it turns out that $|\{u^\beta < k\}| = 0$, then

$$u^\beta(x) \geq k \;\; for \;\; a.e. \;\; x \in \Omega.$$

This ends the proof of Theorem 2.2.
Now we deal with the maximum and minimum principle for weak solution to some elliptic systems.

3 Extremum principle for the weak solutions of elliptic systems

Let Ω be a bounged open subset of R^n, and $u : \Omega \subset R^n \to R^N$; $n, N \geq 2$. We consider the elliptic systems

$$-\sum_{i=1}^{n} D_i(a_i^\alpha(x, u(x)) = 0, \quad x \in \Omega, \quad \alpha = 1, \ldots, N, \tag{3.1}$$

where $a_i^\alpha(x, z) : \Omega \times R^{N\times n} \to R$ is assumed to be measurable with respect to $x \in \Omega$ and continuous with respect to $z \in R^{N\times n}$. We assume that

$$0 < \sum_{i=1}^{n} \sum_{\alpha=1}^{N} a_i^\alpha(x, z)(z_i^\alpha - \tilde{z}_i^\alpha) \tag{3.2}$$

for every $x \in \Omega$, for every couple of matrices $\tilde{z}, z \in R^{N\times n}$, such that there exists $\beta \in \{1, \ldots, N\}$ for which $\tilde{z}^\beta = 0 \neq z^\beta$ and $\tilde{z}^\alpha = z^\alpha$ for $\alpha \neq \beta$.
A function $u \in W^{1,1}(\Omega, R^N)$ is a weak sub(super)-solution to (3.1) if

$$\int_\Omega \sum_{i=1}^{n} \sum_{\alpha=1}^{N} a_i^\alpha(x, u(x) D_i\phi^\alpha(x)dx \leq (\geq) 0, \tag{3.3}$$

for every nonnegative $\phi \in W_0^{1,1}(\Omega, R^N)$.
For $\gamma \in \{1, \ldots, N\}$ and $b \in R$ we define the truncation operator

$$T_{\gamma,b} : R^N \to R^N \tag{3.4}$$

as follows. For every $y = (y^1, \ldots, y^N) \in R^N$ we set $T_{\gamma,b}(y) = (y^1, \ldots, y^\gamma \wedge b, \ldots, y^N)$.
The main result in this section is the following.

Theorem 3.1 *Let $u = (u^1, \ldots, u^N) \in W^{1,1}(\Omega, R^N)$ be a weak sub(super)-solution to (3.1) under (3.2). If there exist $\beta \in \{1, \ldots, N\}$ and $k \in R$ such that*

$$u^\beta \leq (\geq) k \quad on \quad \partial\Omega,$$

then

$$u^\beta \leq (\geq) k \quad in \quad \Omega.$$

Now we prove the Theorem.

Proof of Theorem 3.1. Let $\beta \in \{1, \ldots, N\}$ and $k \in R$ such that $u^\beta \leq (\geq) k$ on $\partial\Omega$. Set

$$\varphi^\beta = -\max(\min)\{u^\beta - k, 0\}.$$

Since $u^\beta \leq (\geq) k$ on $\partial\Omega$, it turns out that $\varphi^\beta \in W_0^{1,1}(\Omega)$. If $\alpha \neq \beta$ we simply set $\varphi^\alpha = 0$. Then we have $\varphi \in W_0^{1,1}(\Omega, R^N)$ and

$$\tilde{u} := u + \varphi \in W_0^{1,1}(\Omega, R^N). \tag{3.5}$$

Set

$$\Omega_1 = \{x \in \Omega : u^\beta \leq (\geq) k\} \cup \{x \in \Omega : u^\beta > (<) k, \quad Du^\beta(x) = 0\}$$

and

$$\Omega_2 = \Omega \setminus \Omega_1.$$

Then

$$D\tilde{u} = Du \quad on \quad \Omega_1 \tag{3.7}$$

$$D\tilde{u}^\alpha = \begin{cases} Du^\alpha, & if \quad \alpha \neq \beta, \\ 0 \neq Du^\beta, & if \quad \alpha = \beta \end{cases} \quad on \quad \Omega_2. \tag{3.8}$$

thus

$$a_i^\alpha(x, \tilde{u}(x)) = a_i^\alpha(x, u(x)) \quad if \quad x \in \Omega_1, \tag{3.9}$$

Using (3.8),(3.9) and "monotonicity" (3.2) we have

$$0 < \int_{\Omega_2} \sum_{i=1}^n a_i^\beta(x, Du)(D_i u^\beta - D_i \tilde{u}^\beta), \tag{3.10}$$

and using (3.3), (3.8) and (3.9) we have

$$\int_{\Omega_2} \sum_{i=1}^n a_i^\beta(x, Du)(D_i u^\beta - D_i \tilde{u}^\beta) \leq 0. \tag{3.11}$$

Comparing (3.10) with (3.11) gives that Ω_2 has zero measure. This means that $Du^\beta(x) = 0$ for almost every $x \in \{u^\beta > (<) k\}$, thus $D\varphi^\beta(x) = 0$ a.e. in Ω. Since $\varphi^\beta \in W_0^{1,1}(\Omega)$, by Poincaré inequality it follows that $\varphi^\beta = 0$ for a.e.

$x \in \Omega$. Since $\varphi^\beta = -\max(\min)\{u^\beta - k, 0\} < (>)0$ on $\{u^\beta > (<)k\}$, it turns out that $|\{u^\beta > (<)k\}| = 0$, then

$$u^\beta(x) \le (\ge) k \quad for \quad a.e. \quad x \in \Omega.$$

This ends the proof of Theorem 3.1.

Remark 3.2 *In the proof of Theorem 3.1, we prove the weak sub-solution to (3.1) satisfies the maximum principle with* $\tilde{u} = T_{\beta,k}(u)$ *and the test function in (3.3) is* $\phi = -\varphi$. *However, we prove the weak super-solution to (3.1) satisfies the minimum principle with* $\tilde{u} = T^{\beta,k}(u)$ *and the test function in (3.3) is* $\phi = \varphi$.

Example 3.3 *Assume that* $a_i^\alpha(x,z) = z_i^\alpha$, *it is easy to have that*

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{N} a_i^\alpha(x,z)(z_i^\alpha - \tilde{z}_i^\alpha) = \sum_{i=1}^{n}\sum_{\alpha=1}(z_i^\alpha)^2 > 0$$

and (3.2) holds true.

Example 3.4 *Assume that* $a_i^\alpha(x,z) = sign(z_i^\alpha)|z|$, *we can obtain*

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{N} a_i^\alpha(x,z)(z_i^\alpha - \tilde{z}_i^\alpha) = \sum_{i=1}^{n}\sum_{\alpha=1}^{N} |z_i^\alpha||z| > 0$$

and (3.2) holds true.

Example 3.5 *Assume that* $a_i^\alpha(x,z) = sign(z_i^\alpha)g(x, |M_1(Du)|, |M_2(Du)|, \ldots, |M_s(Du)|)$, *where* $s = \min\{n, N\}$ *and*

$$p_i \to g(x, p_1, \ldots, p_s)$$

is increasing on $[0, +\infty]$ *for every* $i = 1, \ldots, s$, *see [1,3]. Then (3.2) holds true with* $g(x, p_1, \ldots, p_s) > 0$. *A simple model is*

$$a_i^\alpha(x,z) = sign(z_i^\alpha)(|z|^p - (detz)^q)$$

where $p, q > 0$*; see [2] for the case* $n = 2 = p = q$.

Example 3.6 *For* $a \ge 1/2$, $n = N = 2$, *let us set*

$$a_i^\alpha(x,z) = sign(z_i^\alpha)(a|z|^4 - (detz)^2),$$

then (3.2) simply hold true ,see [1].

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