Mathematica Aeterna, Vol. 6, 2016, no. 6, 815 - 829

CONTROL OF CHAOTIC DYNAMICS BY EXACT LINEARIZATION

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Abstract: This paper deals with an exact linearization approach to control chaos in nonlinear dynamical systems. The linearization technique has been presented as an algorithmic process to stabilize the systems at the control goal. The paper provides a unified framework for studying the two cases of having a point or a limit cycle as the control goal. A few conditions are obtained on the controller parameters that determine whether the control action guides the system asymptotically to a point or a limit cycle. The theoretical results are then applied to Sprott system N to substantiate the effectiveness of this method. The cases of both linear and non-linear output functions are studied. Numerical simulations provide some insight into the geometry of the basins of attraction of the control goal.

Keywords: Chaos Synchronization, Exact linearization, Feedback linearizable, Local diffeomorphsim, Controllable, Goal point, Admissible, Reachable, Local asymptotic stability, Periodic solution, Sprott chaotic system-N

1. Introduction

The word ‘chaos’ defines an aperiodic long term behaviour in deterministic systems that exhibits sensitive dependence on initial conditions. As it arises very frequently in problems of applied sciences, control of chaos has grown into an exceedingly important topic in the study of nonlinear dynamics. Since 1990, wide applications of chaos theory in secret communication has led to heightened interest in this field. Currently, there are several well known methods for controlling chaos[1] such as open loop control, closed loop control, parametric entrainment control, linear, non-linear and adaptive feedback control, etc. Some of these methods stabilize the dynamical systems globally whereas others do so locally, in a neighbourhood of control goal. Generally, the local stabilization methods are rooted in linearizing the systems using Taylor's theorem, which is valid only in a neighbourhood. Apart from these, there are some non-conventional methods of which exact linearizartion control[2] is a prominent example. The idea of converting a non-linear system to a linear system through some suitable non-singular co-ordinate transformation[3] has been proven to be a very powerful method in control theory. Though this method stabilize the system locally, it is often more effective than conventional methods. Notable work in this direction was done by Yu[4] who used input-output linearization method for controlling chaos. In 1998, Kocarev used differential geometric control techniques to non linear dynamical systems[5]. In the subsequent years, Liqun and Yanzhu[6, 7], Alvarez[8] and Tsagas and Mazumdar[9] applied the exact linearization control to chaotic oscillators. Chaos in Chen equations was recently controlled using feedback linearization by Shi and Zhu[10]. Recently Islam et al[11] extended the method to produce a general framework that accommodates both points and limit cycles as the control target. The main theoretical results of this paper are aimed at placing the results found in[11] on a rigorous footing.

Section 2 contains the theoretical results of the paper and Section 3 discusses the application of these results on Sprott system N. Section 2.1 is a recapitulation of the results related to state space exact linearization. Section 2.2 provides rigorous proofs that establish necessary and sufficient condition for stabilizing a chaotic system at a point and a sufficient condition for stabilizing the system onto a limit cycle. Certain definitions are proposed involving goal points, admissibility of goal points and reachability of goal points in connection with the proofs. The resulting algorithmic process to stabilize the system at a point or a on limit cycle is outlined in Section 2.3.

2. Exact linearization and control of chaos

A non-linear dynamical system is generally represented by

$$\dot{x} = f(x) \tag{1}$$

and the corresponding control-affine system by

$$\dot{x} = f(x) + g(x)u \tag{2}$$

where $x \in \mathbb{R}^n$; $f, g \in C^\infty(\mathbb{R}^n)$ and u is a real valued C^∞ function on $\mathbb{R}^n$.

2.1. Exact linearization of non-linear systems.

Definition 1. *A $C^k (k \geq 1)$ function $T : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be a C^k local diffeomorphism if for each $x \in U$, there exists a neighbourhood W_x such that $T|_{W_x}$ is a diffeomorphism.*

Definition 2. *A system of the form* (2) *is said to be feedback linearizable*[1] *with respect to output $y = \lambda(x)$ on an open set $U \subset \mathbb{R}^n$ provided there exists a $C^k (k \geq 1)$ local* *diffeomorphsim $T(x)$ and a smooth function $v = \alpha(x) + \beta(x)u$ such that the coordinate transformation $z = T(x)$ produces a linear controllable system.*

The linear controllable system, in its most general form, is given by :

$$\dot{z} = Cz + Bv, \tag{3}$$

where the pair (C, B) is controllable. It is a well known fact[12] that for feedback linearizable systems, there exists a suitable choice of the local diffeomorphism T such that

$$C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Hence, in what follows, we assume that C and B has the above form.

Definition 3. *Lie bracket of smooth vector fields $F, G : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is defined to be the vector field $[F, G] : U \to \mathbb{R}^n$ given by*

$$[F, G](x) := (DG(x))F(x) - (DF(x))G(x)$$

where $DF(x)$(resp. $DG(x)$) is the derivative of F(resp. G) at the point x.

Let us now introduce the notation

$$\mathrm{ad}_f^k g = [f, \mathrm{ad}_f^{k-1} g], \qquad \text{for all } k \in \mathbb{N}$$

where we define

$$\mathrm{ad}_f^0 g = g.$$

Viewing these as elements in $\mathbb{R}^n$,

$$\mathrm{ad}_f^k g(x) = [(\mathrm{ad}_f^k g(x))_1, (\mathrm{ad}_f^k g(x))_2, \dots, (\mathrm{ad}_f^k g(x))_n]^T,$$

where

$$\left(\mathrm{ad}_f^k g(x)\right)_j = \sum_{i=1}^n \left[f_i \frac{\partial}{\partial x_i} \left(\mathrm{ad}_f^{k-1} g(x)\right)_j - \left(\mathrm{ad}_f^{k-1} g(x)\right)_i \frac{\partial}{\partial x_i} (f_j) \right].$$

With these definitions in hand, we state the two results[12] that are fundamental to exact linearization of non-linear systems.

Result 1. *A system of the form (2) is feedback linearizable in the neighbourhood $N(x_0)$ of $x_0 \in \mathbb{R}^n$ if and only if the following conditions are satisfied on $N(x_0)$:*

1) *The matrix $M = [g(x), ad_f g(x), ..., ad_f^{n-1} g]$ has rank n*
2) *$S = span\{g, ad_f g, ..., ad_f^{n-2} g\}$ is involutive*

If the above two conditions are satisfied, then the existence of the suitable output function $\lambda(x)$ is given by the following result.

Result 2. *If the conditions 1) and 2) in Result 1 hold, then there exists a real valued* $C^k (k \geq 1)$ *function* $\lambda : N(x_0) \to \mathbb{R}$ *such that*

$$L_{ad_f^k g} \lambda(x) = 0, \qquad 0 \leq k \leq n-2$$

and

$$L_{ad_f^{n-1} g} \lambda(x) \neq 0$$

for all $x \in N(x_0)$.

Here we use the notation $L_F G(x)$ to denote the Lie derivative of the real valued function $G(x)$ with respect to the vector field F.

2.2. **Stabilization of the linearized system.** The feedback linearized system (3), with v chosen to be $a_1 z_1 + a_2 z_2 + ... + a_n z_n$ where $a_1, a_2, ..., a_n \in \mathbb{R}$, can be represented by

$$\dot{z} = Az \tag{4}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix}.$$

Let the equilibrium points of this linear system be $N(A) = \{z : Az = 0\}$.

Definition 4. *The admissible set of goal points of system* (2) *under the feedback control* u *is defined to be* $\cup_{z \in N(A)} T^{-1}(z)$ *where* $T^{-1}(z) = \{x : T(x) = z\}$.

Definition 5. *A point* z_0 *is said to be asymptotically reachable from* U *if for all* $z \in U$, *there exists an integral curve of system* (4) *such that* $\lim_{t\to\infty} z(t) = z_0$.

Definition 6. *An admissible goal point is termed a goal point of the system provided it is reachable from an open set* $U(x_0)$ *containing* x_0.

Theorem 1. *A point* x_0 *is a goal point of* (2) *if* $T(x_0)(= z_0) \in N(A)$ *and* z_0 *is an asymptotically stable equilibrium point of* (4).

Proof. To prove this theorem, let us assume that $T(x_o) = z_0 \in N(A)$ and z_0 be an asymptotically stable erquilibrium point of (4). Since z_0 is an asymptotically stable erquilibrium point of (4) and T is a local diffeomorphism, there exist an open set $V(z_0) \in N(z_0)$ such that $T^{-1}(V(z_0)) = U(x_0)$(say) is an open set containing x_0 with an integral curve $\psi(t) \in V(z_0)$ of system (4) such that $\psi(0) = z \in V(z_0)$ and $\lim_{t\to\infty} \psi(t) = z_0$.Therefore $T^{-1}\psi(t) \in U(x_0)$ and $\lim_{t\to\infty} T^{-1}\psi(t) = T^{-1}(\lim_{t\to\infty} \psi(t)) = T^{-1}(z_0) = x_0$.
This shows that $T^{-1}\Psi$ is the required integral curve of system (2) to get x_0 as goal point. □

Analogous to the admissible goal points, it is also possible to treat limit cycles as control goals. Let $C_x(t)$ be a periodic solution of (2). It is then easy to observe that $T(C_x)$ is a periodic solution of (4). The converse holds only when the periodic solution $C_z(t)$ of (4) lies in $T(N(x_0))$, so that $T^{-1}(C_z)$ is well defined. Whenever $T^{-1}(C_z)$ exists and is well defined, it is also periodic as can be observed quite easily.

Theorem 2. *If* (4) *has a periodic solution* C_z *such that* $C_z(t) \in T(N(x_0))$ *for all* $t \geq 0$*, then* $C_x = T^{-1}(C_z)$ *is a periodic solution of* (2)*.*

Proof. Follows from the above discussion. □

Now we restrict our discussions to $\mathbb{R}^3$.

Theorem 3. $T^{-1}(0)$ *is the complete set of goal points if and only if the transformation* $T : N(x_0) \to \mathbb{R}^3$ *produces a system of the form* (4) *with the matrix* A *satisfying* $a_1 < 0, a_2 < 0, a_3 < 0, a_1 + a_2 a_3 > 0$.

Proof. The conditions on A ensure that the equilibrium points of (4) are asymptotically stable. Thus A must be non-singluar, that is, $N(A) = \{0\}$.

By Theorem 1, x_0 is a goal point if and only if $z_0 = T(x_0)$ is asymptotically stable and z_0 is asymptotically stable if and only if A satisfies the inequalities in the theorem. But z_0 can only take the value 0 in this case. Hence, we have, x_0 is a goal point if and only if $T(x_0) = 0$ and A satisfies the conditions of the theorem. □

Theorem 4. *If the matrix* A *in* (4) *satisfies* $a_1 < 0, a_2 < 0, a_3 < 0, a_1 + a_2 a_3 = 0$ *and the periodic solution of* (4) *has sufficiently small amplitude, then* (2) *has a stable periodic solution.*

Proof. If A satisfies the conditions of the theorem, the linear system (4) has a stable periodic solution, say C_z.Further suppose that a_1, a_2 and a_3 are chosen such that C_z has sufficiently small amplitude, that is, $C_z(t) \in T(N(x_0))$ for all $t \geq 0$. Then, by Theorem 2, we have a periodic solution C_x of (2). As C_z is stable and $C_x = T^{-1}(C_z)$, C_x is also stable. □

2.3. Algorithm for control of chaos by exact linearization.

Step 1: : Problem formulation and computation of $\mathrm{ad}_f^k g$

Consider a non-linear dynamical system

$$\dot{x} = f(x) \tag{5}$$

and its corresponding non-linear single input control system as

$$\dot{x} = f(x) + g(x)u, \tag{6}$$

The quantities $\mathrm{ad}_f^k g$ for $0 \leq k \leq n-1$ are computed.

Step 2: : Determination of region where exact linearization is applicable

To find a set Ω such that for all $x_0 \in \Omega$, there exists an open set U_{x_0} such that the matrix

$$M = [g(x), \mathrm{ad}_f g(x), ..., \mathrm{ad}_f^{n-1} g]$$

has rank n and

$$S = \mathrm{span}\{g, \mathrm{ad}_f g, ..., \mathrm{ad}_f^{n-2} g\}$$

is involutive for all $x \in U_{x_0}$.

Step 3: : Determination of output function

Then, by Result 2, there exist a real valued function $\lambda(x)$ in a neighbourhood $N(x_0)$ of the point x_0 such that the following are satisfied :

$$L_g\lambda(x) = L_{ad_f g}\lambda(x) = L_{ad_f^2 g}\lambda(x) = \ldots\ldots = L_{ad_f^{n-2} g} = 0$$

and

$$L_{ad_f^{n-1} g}\lambda(x) \neq 0.$$

The function $\lambda(x)$ is determined by solving the system on $(n-1)$ first order PDEs given by the above conditions.

Step 4: : Determination of the transformation formulae

We have the coordinate transformation $z : N(x_\circ) \to \mathbb{R}^n$ given by,

$$\begin{aligned} z &= (z_1, z_2, \ldots, z_n)^T = T(x) \\ &= [T_1(x), T_2(x), \ldots, T_n(x)]^T \\ &= [\lambda(x), L_f\lambda(x), \ldots, L_f^{n-1}\lambda(x)]^T. \end{aligned} \tag{7}$$

and a smooth transformation of feedback, given by

$$\begin{aligned} v &= \alpha(x) + \beta(x)u \\ &= L_f^n\lambda(x) + L_g L_f^{n-1}\lambda(x)u \end{aligned}$$

By Result 1, with these transformations applied, the non-linear system is transformed to the linear controllable system,

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= v \end{aligned}$$

In order to have a closed loop linear system, let us choose

$$v = a_1 z_1 + a_2 z_2 + \ldots + a_n z_n$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Step 5: : Stabilization of the chaotic system (the specific case n=3)

Firstly, let the matrix A satisfy the conditions $a_1 < 0, a_2 < 0, a_3 < 0, a_1 + a_2 a_3 > 0$. Then, $G = T^{-1}(0)$ gives the set of goal points.

Now let the matrix A satisfy the conditions $a_1 < 0, a_2 < 0, a_3 < 0, a_1 + a_2 a_3 = 0$. Then, we have to choose $x(0)$ such that $z(0)$ is very close to the origin and hence, the resulting limit cycle C_z will be of sufficiently small amplitude. This would stabilize the chaotic system onto the limit cycle C_x.

3. Application of the algorithm to the Sprott chaotic system N

Step 1: : Problem formulation and computation of $\mathrm{ad}_f^k g$

We consider the Sprott chaotic system-N described by

$$\begin{aligned} \dot{x}_1 &= -\alpha x_2 \\ \dot{x}_2 &= x_1 + x_3^2 \\ \dot{x}_3 &= \beta + x_2 - \gamma x_3 \end{aligned} \tag{8}$$

which is chaotic for $\alpha = 2,\ \beta = 1$ and $\gamma = 2$. The above system of equations can be written as

$$\dot{x} = f(x) \tag{9}$$

$$\text{where } \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \text{ and } f(x) = \begin{pmatrix} -\alpha x_2 \\ x_1 + x_3^2 \\ \beta + x_2 - \gamma x_3 \end{pmatrix}.$$

The corresponding nonlinear control system is

$$\dot{x} = f(x) + g(x)u, \tag{10}$$

where $g(x) = (0, 0, -x_3)^T$ and $u(x_1, x_2, x_3)$ is the parametric entrainment control is applied to the parameter γ.

Computation of $\mathrm{ad}_f^k g$ for $k = 1, 2$ gives :

$$\mathrm{ad}_f g(x) = [f, g](x) = \begin{pmatrix} 0 \\ 2x_3^2 \\ -(\beta + x_2) \end{pmatrix}$$

$$\mathrm{ad}_f^2 g(x) = [f, \mathrm{ad}_f g](x) = \begin{pmatrix} 2\alpha x_3^2 + \beta + x_3 \\ 2x_3(3\beta + 3x_2 - 2\gamma x_3) \\ -(\beta + x_2 + \gamma\beta - 2x_3^2) \end{pmatrix}$$

Step 2: : Determination of region where exact linearization is applicable

Here,

$$\begin{aligned} \det(M) &= \begin{vmatrix} 0 & 0 & 2\alpha x_3^2 + \beta + x_3 \\ 0 & 2x_3^2 & 2x_3(3\beta + 3x_2 - 2\gamma x_3) \\ -x_3 & -(\beta + x_2) & -(\beta + x_2 + \gamma\beta - 2x_3^2) \end{vmatrix} \\ &= 2x_3^3(\beta + x_3 + 2\alpha x_3^2) \\ &= 0, \text{if and only if } x_3 = 0. \end{aligned}$$

Let $\Omega = \pi_3^{-1}(\mathbb{R} \setminus \{0\})$ where $\pi_3 : \mathbb{R}^3 \to \mathbb{R}$ is the projection onto the third coordinate. Continuity of π_3 implies that Ω is open. Hence, for any $x_0 \in \Omega$, there exists $U_{x_0} \subset \Omega$, and hence, $\det(M) \neq 0$ for all $x \in U_{x_0}$.

As the third coordinate is nonzero on U_{x_0},

$$[g, \mathrm{ad}_f g](x) = \begin{pmatrix} 0 \\ -4x_3^2 \\ -(\beta + x_2) \end{pmatrix} = \frac{3}{x_3}(\beta + x_2)g(x) + (-2)\mathrm{ad}_f g(x)$$

which establishes that $S = \mathrm{span}\{g(x), \alpha d_f g(x)\}$ is involutive for all $x \in U_{x_0}$.

Thus, exact linearization is applicable on the open subset Ω of $\mathbb{R}^3$.

Step 3, 4 and 5: : Determination of output function, transformation formulae and stabilization of chaos

Solving the system of two PDEs given by $L_g\lambda(x) = 0$ and $L_{\mathrm{ad}_f g}\lambda(x) = 0$, it is observed that $\lambda(x)$ is a function of x_1 only, that is, $\lambda(x) = \psi(x_1)$. It is also necessary to have $L_{\mathrm{ad}_f^2 g}\lambda(x) \neq 0$ on some neighbourhood of x_0, where $x_0 \in \Omega$.

In order to investigate the dependence of the control system on the choice of output function, two separate cases are studied with a linear and a non-linear (quadratic) output function respectively.

Case I : Linear output $\psi(x_1)$

Considering the linear output function$\psi(x_1) = x_1 + x_g$, where $x_g(> 0)$ is a positive real constant, the transformation formulae

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_g \\ -\alpha x_2 \\ -\alpha(x_1 + x_3^2) \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

will transform the chaotic system to the linear controllable system for the control action

$$\begin{aligned} u &= \frac{1}{L_g L_f^2 \lambda(x)}[v - L_f^3\lambda(x)] \\ &= \frac{1}{2\alpha x_3^2}[a_1(x_1 + c) - \alpha a_2 x_2 - \alpha a_3(x_1 + x_3^2) - \alpha^2 x_2 + 2\alpha x_3(\beta + x_2 - \gamma x_3)] \end{aligned}$$

where $a_1, a_2, a_3 \in \mathbb{R}$ are the control parameters. The inverse transformation is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = T^{-1}(Z) = \begin{pmatrix} T_1^{-1}(z) \\ T_2^{-1}(z) \\ T_3^{-1}(z) \end{pmatrix} = \begin{pmatrix} z_1 - x_g \\ -\frac{1}{\alpha}z_2 \\ \pm\sqrt{-\frac{1}{\alpha}z_3 - z_1 + x_g} \end{pmatrix}$$

The set of goal points, G is given by $G = \{(-x_g, 0, \pm\sqrt{x_g}) : x_g > 0\}$. Suppose a_1, a_2, a_3 have been chosen suitably. Then, modifying x_g, we can drive the system towards any goal point in G. In this case, any point on the curve $x_3^2 = x_1, x_2 = 0$ can be reached with this control action u.

Simulation results and discussion for Case I :

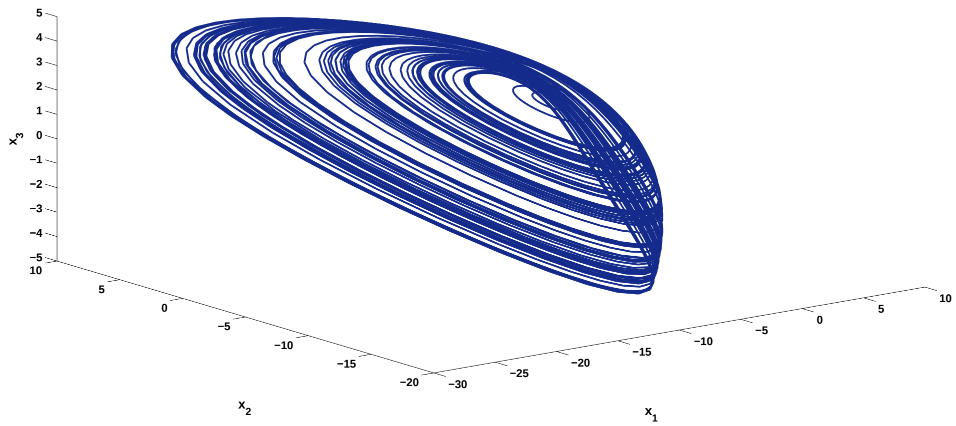

FIGURE 1. Phase portrait of Sprott-N chaotic system

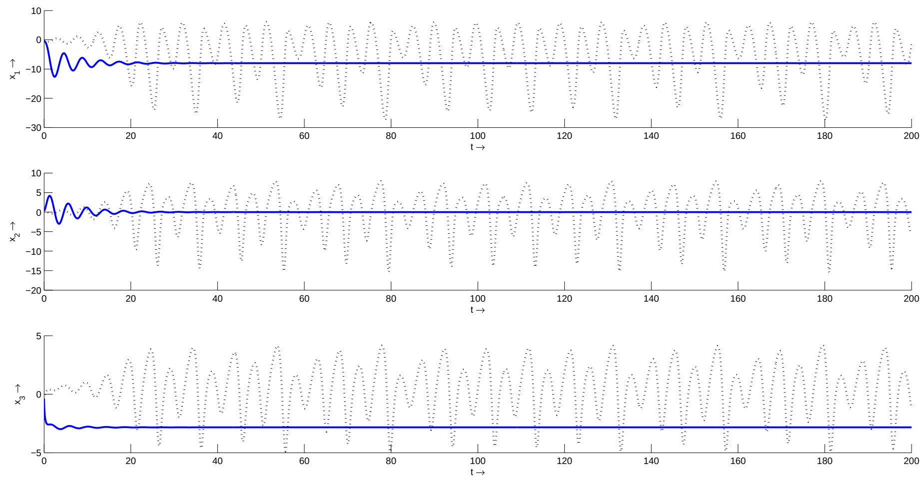

FIGURE 2. Stabilization of the state trajectories

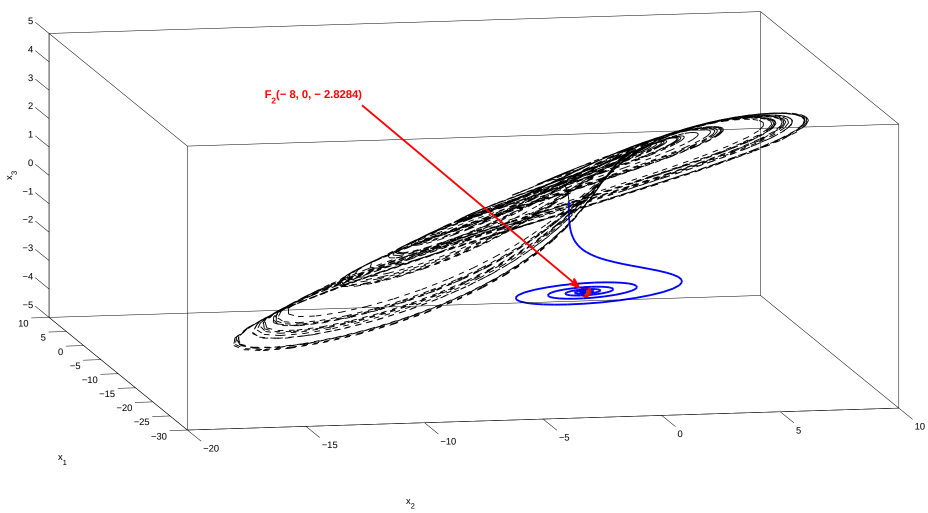

FIGURE 3. Convergence of the chaotic system to the control goal

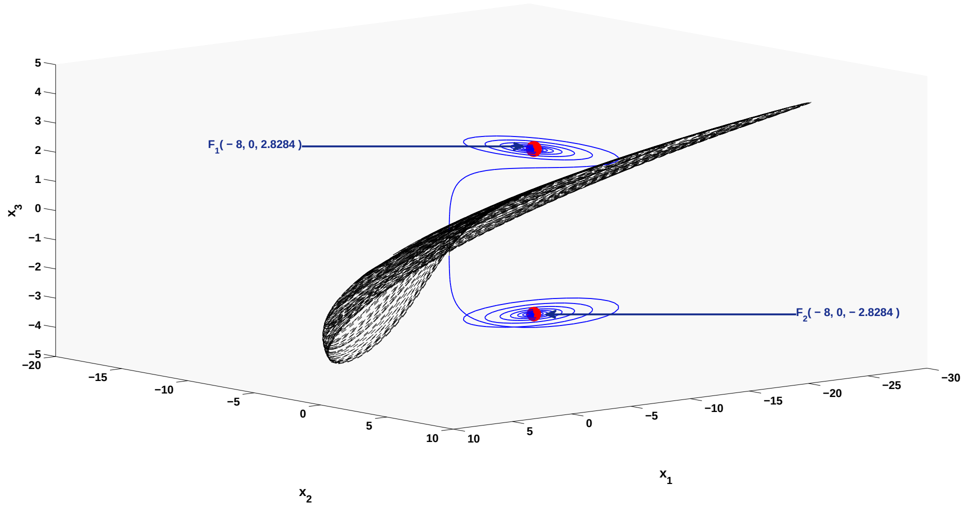

FIGURE 4. Convergence to the control goals separated by chaotic attractor

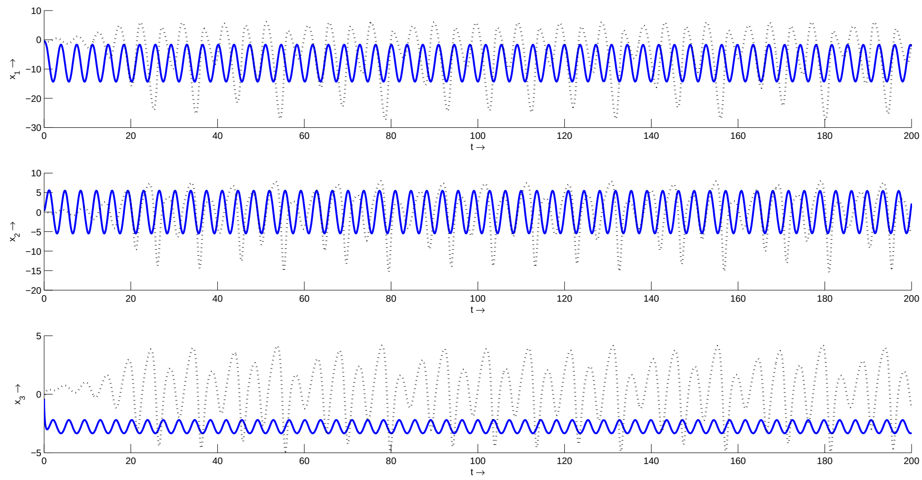

FIGURE 5. Stabilization of the state trajectories to periodic motion for limit cycles

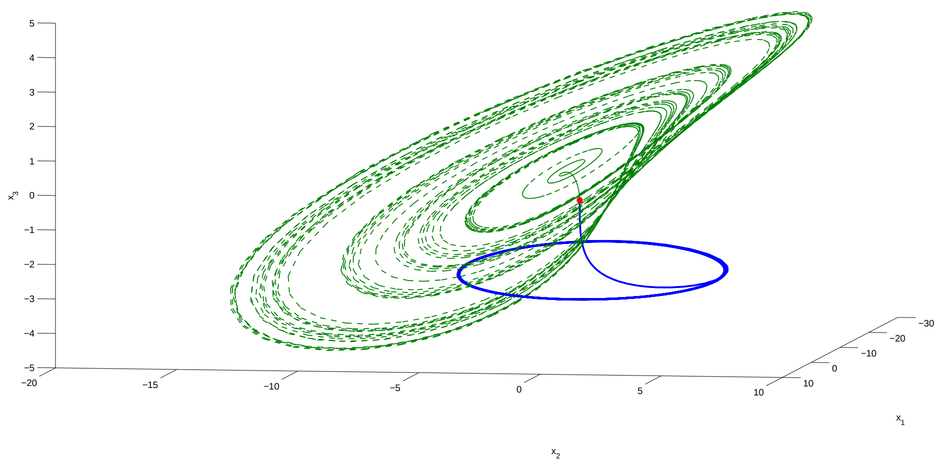

FIGURE 6. Convergence of the chaotic system to the limit cycle

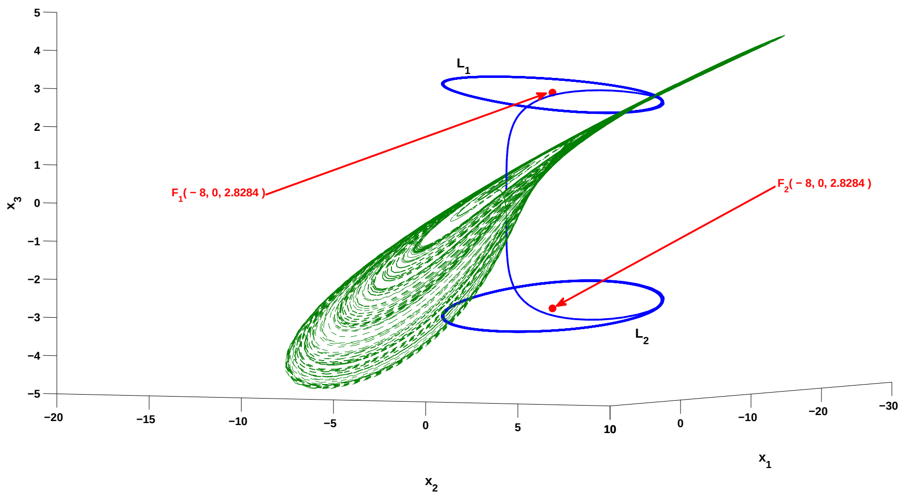

FIGURE 7. Convergence of the chaotic system to the limit cycles separated by attractor

Figure 1 shows the phase portrait of the Sprott chaotic system-N with parameter values $\alpha = 2, \beta = 1$ and $\gamma = 2$. In figure 2, we present the change of the state variables x_1, x_2 and x_3 with varying time t. The dotted line represents the behaviour of the state variables of the original non linear chaotic system whereas the solid line gives the same for the controlled chaotic system. In figure 3 and figure 4, the control parameters are given the value $a_1 = -6, a_2 = -3$ and $a_3 = -3$ so that the matrix A has only negative eigenvalues. For figure 3, we have the initial conditions $x_1(0) = -0.4, x_2(0) = 0.4$ and $x_3(0) = 0.4$, lying in the region corresponding to x_3 positive. It is observed that for this choice of initial condition, the trajectories asymptotically settle down to the goal point $F_2 = (-8, 0, -2.8248)$ corresponding to $x_g = 8$. If a new intial condition is chosen by taking x_3 to be the negative of the previous value and rest of the coordinates fixed, that is, with initial conditions $x_1(0) = -0.4, x_2(0) = 0.4$ and $x_3(0) = -0.4$, it is seen that the system stabilizes at $F_1 = (-8, 0, 2.8248)$. F_1 and F_2 are separated by the chaotic attractor itself,as seen in Figure 4. The essence of Figure 4 is that it gives us a good geometric idea of how the choice of initial condition will determine what state the system will ultimately reside in. Given a fixed $x_g > 0$, there are two goal points F_1 and F_2, given by $(x_g, 0, \sqrt{x_g})$ and $(x_g, 0, -\sqrt{x_g})$ respectively. Let $N(F_1)$ and $N(F_2)$ be the respective neighbourhoods where the system is exactly linearizable. Here, $x_g \neq 0$ as the exact linearization carried out in this problem holds only for $x_3 \neq 0$. Clearly, there exists non-empty disjoint open sets U_1 and U_2 such that $F_1 \in U_1$ and $F_2 \in U_2$. If we define $V_i = U_i \cap N(F_i)$ for $i = 1, 2$, then for any $x \in V_i$, the trajectories of the system starting at x asymptotically reach $F_i (i = 1, 2)$.

With open sets $V_i^{'} \subset N(F_i)$ (defined with the same motivation as above) and the control parameters modified to $a_1 = -9, a_2 = -3$ and $a_3 = -3$, we obtain analogous results for the goal cycles L_i, where $i = 1, 2$. For initial conditions in $V_1^{'}$, sufficiently close to F_1, with the x_3 coordinate positive, the system settles down onto the goal cycle L_1 around F_1.

Similarly, x_3 negative causes the system to reach the limit cycle L_2. Figure 7 illustrates these results. Figure 5 gives time series of the state variables in this situation. Figure 6 displays the chaotic attractor and the limit cycle together.

Case 2 : Quadratic output $\psi(x_1)$

Let the quadratic output be $\psi(x_1) = x_1^2 - x_g$, where $x_g(> 0)$ is a positive constant.

The transformation formulae

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_1^2 - x_g \\ -2\alpha x_1 x_2 \\ 2\alpha^2 x_2^2 - 2\alpha x_1^2 - 2\alpha x_1 x_3^2 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

will transform the chaotic system to the linear controllable system for the control parameter

$$\begin{aligned} u &= \frac{1}{L_g L_f^2 \lambda(x)}[v - L_f^3 \lambda(x)] \\ &= \frac{1}{4\alpha x_1 x_3^2}[a_1(x_1^2 - x_g) - 2\alpha a_2 x_1 x_2 + a_3(2\alpha^2 x_2^2 - 2\alpha x_1^2 - 2\alpha x_1 x_3^2) - 6\alpha^2 x_2(x_1 + x_3^2) \\ &\quad - 2\alpha^2 x_1 x_2 - 4\alpha^2 x_2(x_1 + x_3^2) + 4\alpha x_1 x_3(\beta + x_2 - \gamma x_3)] \end{aligned}$$

with $a_1, a_2, a_3 \in \mathbb{R}$.

The inverse transformation is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = T^{-1}(Z) = \begin{pmatrix} T_1^{-1}(z) \\ T_2^{-1}(z) \\ T_3^{-1}(z) \end{pmatrix} = \begin{pmatrix} \pm\sqrt{z_1 + x_g} \\ \mp\frac{z_2}{2\alpha\sqrt{z_1+x_g}} \\ \pm\sqrt{\frac{1}{\pm 2\alpha\sqrt{z_1+x_g}}[\frac{z_2^2}{2(z_1+x_g)} - 2\alpha(z_1 + x_g) - z_3]} \end{pmatrix}$$

The set of goal points G is given by $G = \{(-\sqrt{x_g}, 0, \pm x_g^{\frac{1}{4}}) : x_g > 0\}$. Suppose a_1, a_2, a_3 has been chosen as discussed in the paper. Then modifying x_g, we can drive the system to any goal point in G. Again, we can choose a_1, a_2, a_3 such that it reaches a goal cycle. Then, modifying x_g, we can drive the system to a limit cycle around any point in G.

Simulation results and discussion for Case II :

The observations of Case II are almost similar to those made in Case I. The standard initial condition used for all the figures are $x_g = 0.4$ and $x_1 = -0.4, x_2 = 0.4, x_3(0) = 0.4$. In order to illustrate how the control goal makes a jump with specific changes in initial condition, the initial condition is changed to $x_1 = -0.4, x_2 = 0.4, x_3(0) = -0.4$. Figure 8 and 9 gives the time series and the phase diagram of the system for the case of convergence to a goal point. Here, the control parameters were chosen as $a_1 = -12, a_2 = -4, a_3 = -4$. Figure 11 and 12 provide the time series and phase diagrams concerning convergence to a goal cycle, obtained with control parameters $a_1 = -16, a_2 = -4, a_3 = -4$. Figures 10 and 13 are important as they reveal the basins of attraction of steady states and periodic states of the controlled system to a certain extent. Given x_g, let the two possible goal points be given by E_1 and E_2. Following the discussion for simulation in Case I, here we can again define open sets $W_i \subset N(E_i)$ such that for all $x \in W_i$, the trajectories of the starting at

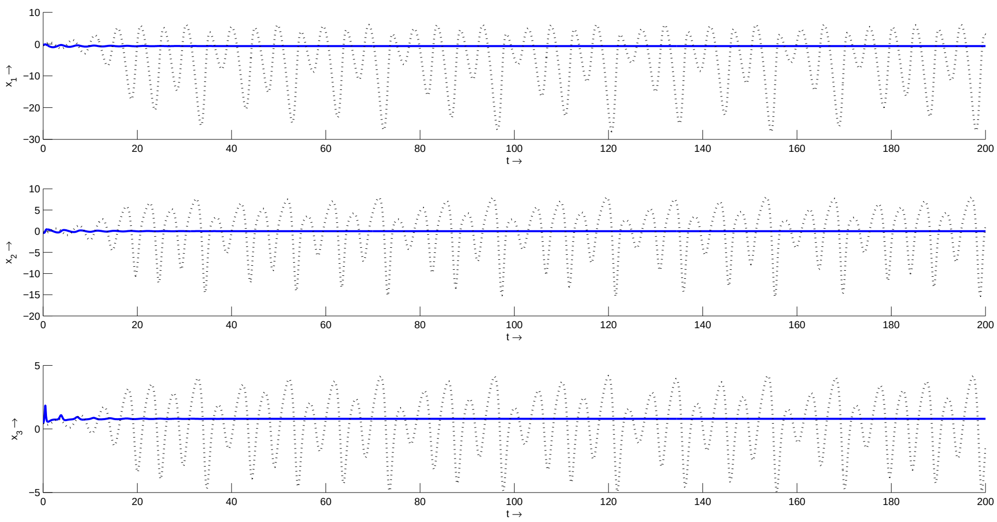

FIGURE 8. Stabilization of the state trajectories for the quadratic output function

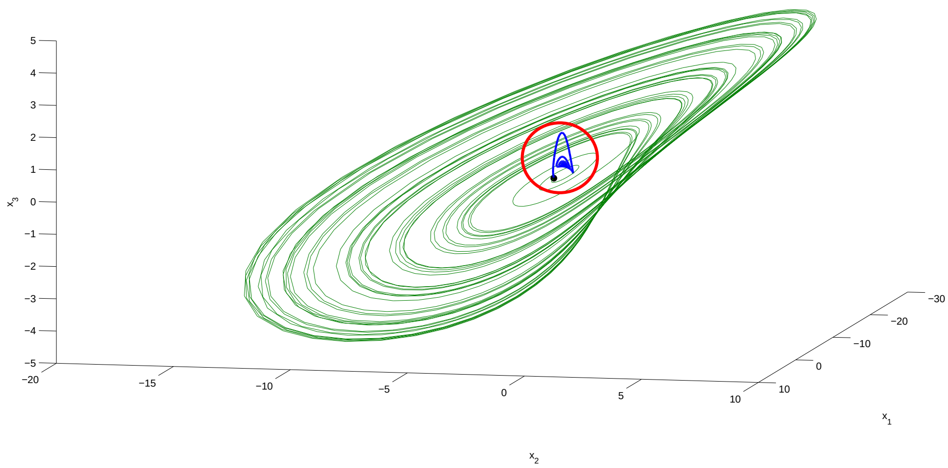

FIGURE 9. Convergence of the chaotic system to the control goal for quadratic output

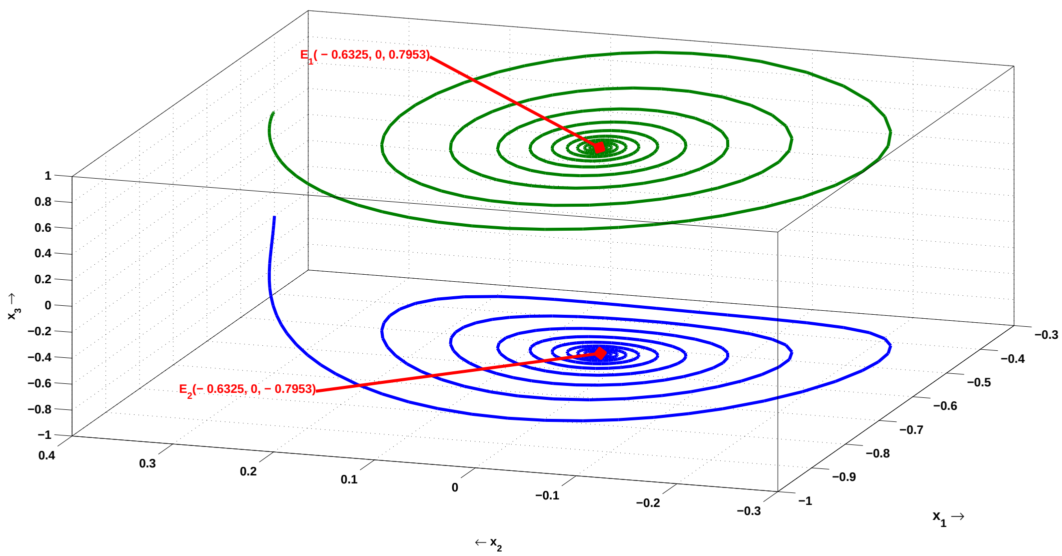

FIGURE 10. Convergence to the control goals for quadratic output

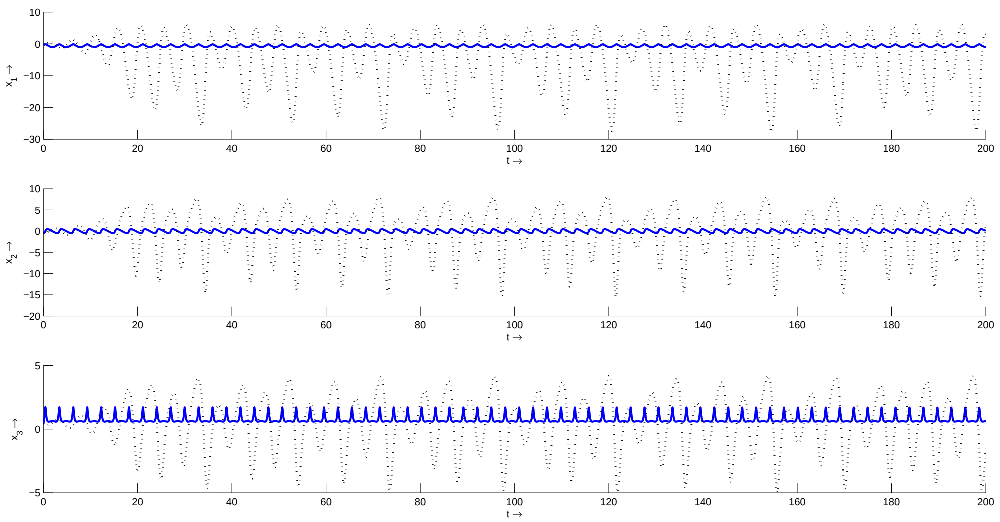

FIGURE 11. Stabilization of the state trajectories to periodic motion for limit cycles in quadratic output

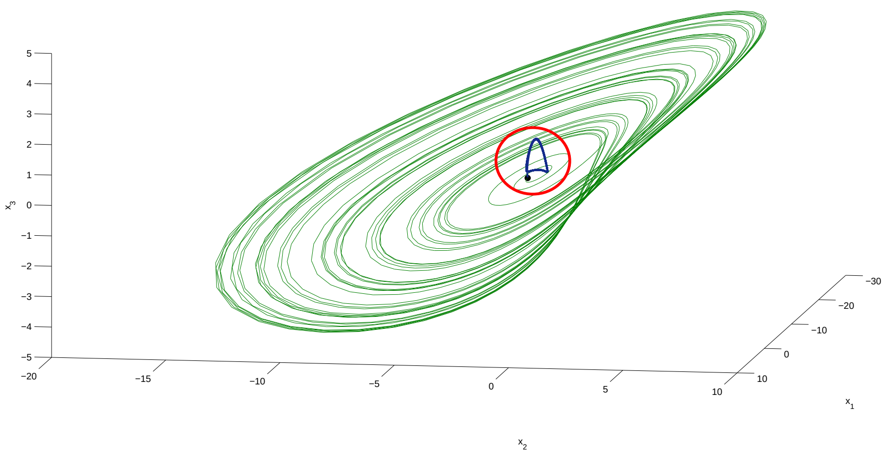

FIGURE 12. Convergence of the chaotic system to the limit cycle for quadratic output

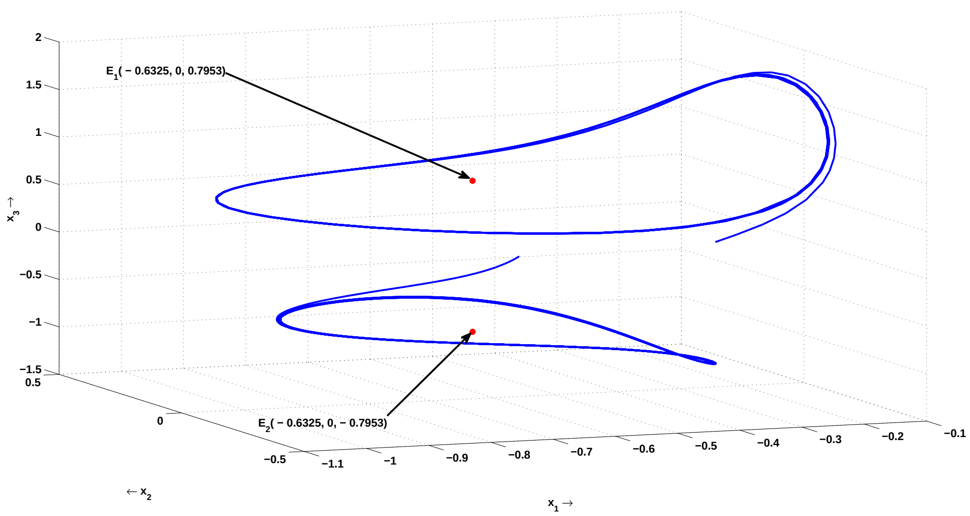

FIGURE 13. Convergence of the chaotic system to the limit cycles for quadratic output

x asymptotically reach E_i $(i = 1, 2)$. This has been shown in Figure 10. For a fixed x_g, the system also admits two possible goal cycles. There exists open sets $W_i^{'} \subset N(E_i)$ such that for any $x \in W_i^{'}$, the trajectory starting at x settles down to a periodic state, that is, a goal cycle around the point $E_i(i = 1, 2)$. Figure 13 is an illustration of this observation.

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Received: November 12, 2016