Mathematica Aeterna, Vol. 6, 2016, no. 6, 923 - 932

On (p, q)-Fibonacci octonions

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Abstract

In this paper, we aim at establishing some formulas and identities for a new class of octonions called the (p, q)-Fibonacci octonions which is introduced here.

Mathematics Subject Classification: 11B39, 05A15, 11R52.

Keywords: Fibonacci numbers, Lucas numbers, quaternions, octonions.

1 Introduction

Fibonacci and Lucas quaternions and octonions are another important step in the develoment of Fibonacci and Lucas numbers theory.
We deal here with the algebra of quaternions over $\mathbb{R}$-denoted by $\mathbb{H}$ with the canonical basis, $\{1 \simeq e_0, i \simeq e_1, j \simeq e_2, k \simeq e_3\}$ having the multiplication rules

in tabular form:

$\times$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	$-e_2$
e_2	e_2	$-e_3$	-1	e_1
e_3	e_3	e_2	$-e_1$	-1

A quaternion is a element of $\mathbb{H}$, and a quaternion is defined by

$$\alpha = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3\ e_3, a_i \in \mathbb{R}, i = 0, 1, 2, 3$$

(see, [3]). For the first time Horadam [6] introduced and studied the so-called Fibonacci and Lucas quaternions, which are new classes of quaternion numbers for the classic Fibonacci and Lucas numbers. They are given respectively by the following recurrence relations:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$

and

$$T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3},$$

where F_n and L_n, respectively, are the nth classic Fibonacci and Lucas numbers that are given respectively by the following recurrence relations for $n \geq 0$:

$$F_{n+2} = F_{n+1} + F_n$$

and

$$L_{n+2} = L_{n+1} + L_n$$

with the initial values $F_0 = 0, F_1 = 1, L_1 = 2$ and $L_1 = 1$ (see, [11]).
Fibonacci quaternions and their generalizations have been presented and studied in the several papers (see, [1], [2], [4], [5], [6], [7], [8], [12], [13]).

The octonions in Clifford algebra **C** are a normed division algebra with eight dimensions over the real numbers larger than the quaternions. The field $\mathbb{O} \cong \mathbb{C}^4$ of octonions

$$\alpha = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3\ e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7, a_i(i = 0, 1, ..., 7) \in \mathbb{R}$$

is an eight-dimensional non-commutative and non-associative $\mathbb{R}$-field generated by eight base elements $e_0, e_1, ..., e_6$ and e_7. The multiplication rules for the basis of $\mathbb{O}$ are listed in the following table[14]:

$\times$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

(1)

We refer the reader to [3] for quaternions and octonions.
Keçilioğlı ve Akkuş [10] introduced Fibonacci and Lucas octonions and gave some identities and properties of them. They are given respectively by the following recurrence relations:

$$Q_n = \sum_{s=0}^{7} F_{n+s} e_s, \tag{2}$$

and

$$T_n = \sum_{s=0}^{7} L_{n+s} e_s,$$

where F_n and L_n, respectively, are the nth classic Fibonacci and Lucas numbers.

The main purpose of the present paper is to give a very wide generalization called the (p,q)-*Fibonacci octonion sequence* $\{\mathbf{O}_n(p,q)\}_{n\geq 0}$ of the Fibonacci octonion sequence given by (2), and then to obtain new and interesting formulas and identities involving the sequence $\{\mathbf{O}_n(p,q)\}_{n\geq 0}$.

Our paper is organized as follows: the main results and their proofs for (p,q)-Fibonacci octonions is stated in the next section. Conclusions are presented in the last section.

2 (p,q)-Fibonacci Octonions

A generalization of the classic Fibonacci sequence $\{F_n\}_{n\geq 0}$ which are called the (p,q)-Fibonacci sequence $F_n(p,q)\}_{n\geq 0}$ is defined by the following recurrence relation for $p^2+4q>0$ and $n \geq 0$:

$$F_{n+2}(p,q) = pF_{n+1}(p,q) + qF_n(p,q) \tag{3}$$

with $F_0(p,q)=0$ and $F_1(p,q)=1$. The paper [8] was devoted to studying the following quaternionic sequence for $n \geq 0$:

$$\mathcal{Q}F_n(p,q) = F_n e_0 + F_{n+1} e_1 + F_{n+2} e_2 + F_{n+3} e_3$$

where F_n is the nth (p,q)-Fibonacci number and e_0, e_1, e_2, e_3 is the basis in $\mathbb{H}$.

Definition 2.1 *The (p,q)-Fibonacci octonion sequence $\{\mathbf{O}_n(p,q)\}_{n\geq 0}$ is defined by the following recurrence relation:*

$$\mathbf{O}_n(p,q) = \sum_{s=0}^{7} F_{n+s} e_s \tag{4}$$

where F_n is the nth generalized (p,q)-Fibonacci number.

Before proceeding to the study of the (p,q)-Fibonacci octonion sequence, we fix the following prpperties which will useful in our computations.

1. The characteristic equation of (3) is
$$x^2 - px - q = 0. \tag{5}$$

2. Solving this equation for $p^2 + 4q > 0$, we get two distinct characteristic roots:
$$\gamma = \frac{p+\sqrt{\Delta}}{2}, \delta = \frac{p-\sqrt{\Delta}}{2},$$
where $\Delta = p^2 + 4q$.

3. Binet's formula for the sequence $F_n(p,q)\}_{n\geq 0}$ is
$$F_n(p,q) == \frac{\gamma^n - \delta^n}{\gamma - \delta}. \tag{6}$$

4. For $p^2 + 4q > 0$, the numbers γ and δ are real and $\gamma \neq \delta$. Also notice that
$$\gamma^2 + q = \gamma\sqrt{\Delta} \tag{7}$$
and
$$\delta^2 + q = -\delta\sqrt{\Delta}. \tag{8}$$

5. For every non-negative integer m
$$(a+b)^m = \sum_{n=0}^{m} \binom{m}{n} a^n b^{m-n}$$
where a and b are any real numbers.

These properties will be used extensively in the proofs of our main results.

From the definitions of (3) and (4), we obtain

$$\mathbf{O}_{n+1} = p\mathbf{O}_n + qQ_{n-1} \tag{9}$$

for $n \geq 1$.

Theorem 2.2 *Let $\mathbf{O}_n$ be the nth (p,q)-Fibonacci octonion number. Then*

$$\mathbf{O}_n = \frac{\underline{\gamma}\gamma^n - \underline{\delta}\delta^n}{\gamma - \delta}, \tag{10}$$

where $\underline{\gamma} = \sum_{s=0}^{7} \gamma^s e_s$ and $\underline{\delta} = \sum_{s=0}^{7} \delta^s e_s$.

Proof 2.3 *From* (4) *and* (6), *we have (10) with*

$$\mathbf{O}_n = \sum_{s=0}^{7} \left(\frac{\gamma^{n+s} - \delta^{n+s}}{\gamma - \delta} \right) e_s.$$

in which $\underline{\gamma} = \sum_{s=0}^{7} \gamma^s e_s$ and $\underline{\delta} = \sum_{s=0}^{7} \delta^s e_s$.

It is well known that for $\mathbf{O}_n$ defined by (4) the ordinary generating function is $G(x) = \sum_{n=0}^{\infty} \mathbf{O}_n x^n$ and the exponential generating function is $E(x) = \sum_{n=0}^{\infty} \mathbf{O}_n \frac{x^n}{n!}$.

Theorem 2.4 *For $\mathbf{O}_n$ defined by* (4), *we have:*

$$G(x) = \frac{\mathbf{O}_0 + (-p\mathbf{O}_0 + \mathbf{O}_1)}{1 - px - qx^2}. \tag{11}$$

Proof 2.5 *Let $G(x) = \sum_{n=0}^{\infty} \mathbf{O}_n x^n$. Substituting the recurrence relation* (9) *into* $(1 - px - qx^2)\, G(x)$ *and after some lengthy manipulation, we have* (11).

Theorem 2.6 *For $\mathbf{O}_n$ defined by* (4), *we have:*

$$E(x) = \frac{\underline{\gamma} e^{\gamma x} - \underline{\delta} e^{\delta x}}{\gamma - \delta}.$$

Proof 2.7 *Using* (10) *in* $E(x) = \sum\limits_{n=0}^{\infty} \mathbf{O}_n \frac{x^n}{n!}$, *we obtain:*

$$E(x) = \sum_{n=0}^{\infty} (\frac{\underline{\gamma}\gamma^n - \underline{\delta}\delta^n}{\gamma - \delta}) \frac{x^n}{n!}. \tag{12}$$

Now we end the proof by combining $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ *and* (12)*.*

Theorem 2.8 *Let* m *be a non-negative integer. Then*

$$\sum_{n=0}^{m} \binom{m}{n} \mathbf{O}_{2n+k} q^{m-n} = \begin{cases} \mathbf{O}_{k+m} \Delta^{\frac{m}{2}}, & meven \\ \mathbf{O}_{k+m} \Delta^{\frac{m-1}{2}}, & modd \end{cases} . \tag{13}$$

Proof 2.9 *Let the left-hand side of the assertion* (13) *of Theorem 2.8 be denoted by* S_1*. From* (10)*, we have*

$$S_1 = \sum_{n=0}^{m} \binom{m}{n} (\frac{\underline{\gamma}\gamma^{2n+k} - \underline{\delta}\delta^{2n+k}}{\gamma - \delta}) q^{m-n}.$$

Note that $\sum\limits_{n=0}^{m} \binom{m}{n} (\gamma^2)^n q^{m-n} = (\gamma^2 + q)^m$ *and* $\sum\limits_{n=0}^{m} \binom{m}{n} (\delta^2)^n q^{m-n} = (\delta^2 + q)^m$. *Combining this with* (7) *and* (8) *we get that*

$$S_1 = \frac{\underline{\gamma}\gamma^k}{\gamma - \delta} (\gamma\sqrt{\Delta})^m - \frac{\underline{\delta}\delta^k}{\gamma - \delta} (-\delta\sqrt{\Delta})^m.$$

If m *is even, then*

$$S_1 = (\frac{\underline{\gamma}\gamma^{k+m} - \underline{\delta}\delta^{k+m}}{\gamma - \delta}) \Delta^{\frac{m}{2}}$$

and hence

$$S_1 = \mathbf{O}_{k+m} \Delta^{\frac{m}{2}}.$$

If m *is odd, then*

$$S_1 = \left(\underline{\gamma}\gamma^{k+m} + \underline{\delta}\delta^{k+m}\right) \Delta^{\frac{m-1}{2}} \tag{14}$$

since $\gamma - \delta = \sqrt{\Delta}$*. Finally, if we apply the following the Binet formula for the nth* (p, q)*-Lucas octonion number* $\mathbf{K}_n$:

$$\mathbf{K}_n = \underline{\gamma}\gamma^n + \underline{\delta}\delta^n [7]$$

for evaluating the right-hand side in (14) *we arrive at the desired result* (13) *for any odd integer* m*.*

Theorem 2.10 *Let m be a non-negative integer. Then*

$$\sum_{n=0}^{m}\binom{m}{n}(-1)^{n}\,\mathbf{O}_{2n+k}q^{m-n}=\begin{cases} p^{m}\mathbf{O}_{k+m}, & p even \\ -p^{m}\mathbf{O}_{k+m}, & p odd \end{cases}. \tag{15}$$

Proof 2.11 *For convenience, let the left-hand side of the assertion (15) of Theorem 2.10 be denoted by S_2. Applying (10), we have that*

$$S_2=\sum_{n=0}^{m}\binom{m}{n}(-1)^{n}\,(\frac{\underline{\gamma}\gamma^{2n+k}-\underline{\delta}\delta^{2n+k}}{\gamma-\delta})q^{m-n}. \tag{16}$$

Employing $\sum\limits_{n=0}^{m}\binom{m}{n}(-\gamma^2)^n q^{m-n}=(-\gamma^2+q)^m$ and $\sum\limits_{n=0}^{m}\binom{m}{n}(-\delta^2)^n q^{m-n}=(-\delta^2+q)^m$ into we get that in this case

$$S_2=\frac{\underline{\gamma}\gamma^{k}}{\gamma-\delta}(-\gamma^2+q)^m-\frac{\underline{\delta}\delta^{k}}{\gamma-\delta}(-\delta^2+q)^m. \tag{17}$$

We know by the characteristic equation in (5) that the roots of this equation can be written as $-p\gamma=-\gamma^2+q$ and $-p\delta=-\delta^2+q$. Inserting these into (17) gives

$$\begin{aligned} S_2 &= (-p)^m(\frac{\underline{\gamma}\gamma^{k+m}-\underline{\delta}\delta^{k+m}}{\gamma-\delta}) \\ &= (-p)^m\mathbf{O}_{k+m}. \end{aligned}$$

Thus, we complete the proof.

Theorem 2.12 *Let m be a non-negative integer. Then*

$$\sum_{n=0}^{m}\binom{m}{n}p^{n}\mathbf{O}_{n}q^{m-n}=\mathbf{O}_{2m}. \tag{18}$$

Proof 2.13 *Let us denote $S_3=\sum\limits_{n=0}^{m}\binom{m}{n}p^n\mathbf{O}_n q^{m-n}$. Applying the Binet formula (10) we transform the left-hand side of (18) into:*

$$S_3=\sum_{n=0}^{m}\binom{m}{n}p^{n}(\frac{\underline{\gamma}\gamma^{n}-\underline{\delta}\delta^{n}}{\gamma-\delta})q^{m-n}.$$

With elementary calculations we have that:

$$S_3=\frac{\underline{\gamma}}{\gamma-\delta}\sum_{n=0}^{m}\binom{m}{n}(p\gamma)^{n}\,q^{m-n}-\frac{\underline{\delta}}{\gamma-\delta}\sum_{n=0}^{m}\binom{m}{n}(p\delta)^{n}\,q^{m-n}.$$

We can now use $\sum_{n=0}^{m}\binom{m}{n}(p\gamma)^n q^{m-n} = (p\gamma+q)^m$ *and* $\sum_{n=0}^{m}\binom{m}{n}(p\delta)^n q^{m-n} = (p\delta+q)^m$ *to conclude that*

$$S_3 = \frac{\underline{\gamma}\gamma^{2m} - \underline{\delta}\delta^{2m}}{\gamma-\delta}$$

which completes the proof of Theorem 2.12.

Theorem 2.14 *Let m be a non-negative integer. Then*

$$\sum_{n=0}^{m}\binom{m}{n}(\mathbf{O}_n)^2 q^{m-n} = \begin{cases} \left(\underline{\gamma}^2\gamma^m + \underline{\delta}^2\delta^m\right)\Delta^{\frac{m-2}{2}}, & meven \\ \left(\underline{\gamma}^2\gamma^m - \underline{\delta}^2\delta^m\right)\Delta^{\frac{m-2}{2}} & modd \end{cases}.$$

Proof 2.15 *Let us denote* $S_4 = \sum_{n=0}^{m}\binom{m}{n}(\mathbf{O}_n)^2 q^{m-n}$. *It follows from* (10) *that the sum* S_4 *can be written in a concise form in terms of the roots of Eq.* (5) :

$$S_4 = \sum_{n=0}^{m}\binom{m}{n}\left(\frac{\underline{\gamma}\gamma^n - \underline{\delta}\delta^n}{\gamma-\delta}\right)^2 q^{m-n}$$

or

$$\begin{aligned} S_4 &= \frac{\underline{\gamma}^2}{(\gamma-\delta)^2}\sum_{n=0}^{m}\binom{m}{n}\left(\gamma^2\right)^n q^{m-n} + \frac{\underline{\delta}^2}{(\gamma-\delta)^2}\sum_{n=0}^{m}\binom{m}{n}\left(\delta^2\right)^n q^{m-n} \qquad (19) \\ &\quad - \frac{\left(\gamma\underline{\delta}+\underline{\delta\gamma}\right)}{(\gamma-\delta)^2}\sum_{n=0}^{m}\binom{m}{n}(\gamma\delta)^n q^{m-n}. \end{aligned}$$

The sums $\sum_{n=0}^{m}\binom{m}{n}\left(\gamma^2\right)^n q^{m-n}$ *and* $\sum_{n=0}^{m}\binom{m}{n}\left(\delta^2\right)^n q^{m-n}$ *are respectively equal to*

$$\sum_{n=0}^{m}\binom{m}{n}\left(\gamma^2\right)^n q^{m-n} = \left(\gamma^2+q\right)^m \qquad (20)$$

and

$$\sum_{n=0}^{m}\binom{m}{n}\left(\delta^2\right)^n q^{m-n} = \left(\delta^2+q\right)^m. \qquad (21)$$

We use (19), (20) *and* (21) *with* (7), (8) *and* $\gamma\delta = -q$ *to obtain*

$$S_4 = \frac{\underline{\gamma}^2\left(\gamma\sqrt{\Delta}\right)^m + \underline{\delta}^2\left(-\delta\sqrt{\Delta}\right)^m}{(\gamma-\delta)^2}. \qquad (22)$$

If m is even, the equality (22) *becomes the following formula*

$$S_4 = \left(\underline{\gamma}^2\gamma^m + \underline{\delta}^2\delta^m\right)\Delta^{\frac{m-2}{2}}.$$

Similarly, if m is odd, the equality (22) *becomes*

$$S_4 = \left(\underline{\gamma}^2\gamma^m - \underline{\delta}^2\delta^m\right)\Delta^{\frac{m-2}{2}}.$$

3 Conclusions

In this work, we introduced and studied some fundamental properties and characteristics of the (p,q)-Fibonacci octonion sequence.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgements

We would like to thank the editor and the reviewers for their helpful comments on our manuscript.

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Received: December 06, 2016