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Some results on Lorentzian α-Sasakian manifold

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Abstract

The aim of the present paper is to study projective pseudosymmetric, concircular pseudo symmetric, generalized projective ϕ-recurrent and generalized concircular ϕ-recurrent conditions on a Lorentzian α-Sasakian manifold.

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1 Introduction

In [23], Yildiz and Murathan by considering α-Sasakian structure as a special case of trans-Sasakian structure of type (α, β), investigated Lorentzian α-Sasakian manifold. Here they proved that conformally flat and quasi conformally flat Lorentzian α-Sasakian manifold is isometric to the sphere $S^{2n+1}(c)$, where $c = \alpha^2$. Different curvature properties of Lorentzian α-Sasakian manifold was studied by many geometers [[2], [14], [21], [22]].

The concept of a pseudosymmetric manifold was introduced by Chaki [5] and Deszcz [7] in two different ways. The two types of pseudosymmetric manifolds are different in their nature. A Riemannian manifold (M^n, g) for $n \geq 3$

is said to be pseudosymmetric according to Deszcz [7] if at every point of M^n the curvature tensor satisfies $R(X,Y)\cdot R = L\,[(X\wedge Y)\cdot R]$, where the dot means that $R(X,Y)$ acts as a derivation on R, L is a smooth function and the endomorphism $X\wedge Y$ is defined by

$$(X\wedge Y)Z = g(Y,Z)X - g(Z,X)Y, \tag{1}$$

for all vectors fields X, Y, Z on M^n.

The notion of generalized recurrent manifolds was introduced by De and Guha [6]. A Riemannian manifold (M^n, g) is called generalized recurrent if its curvature tensor R satisfies the condition

$$(\nabla_W R)(X,Y)Z = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \tag{2}$$

where A and B are two 1-forms and they are defined by $A(X) = g(X,\rho_1)$, $B(X) = g(X,\rho_2)$, and ρ_1, ρ_2 are vector fields associated with 1-forms A, B respectively.

Analogously to the consideration of generalized recurrent manifolds, Prakasha and Yildiz in [15] given the following definition on Lorentzian α-Sasakian manifold.

A Lorentzian α-Sasakian manifold is said to be generalized ϕ-recurrent if its curvature tensor R satisfies the condition

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \tag{3}$$

where A and B are two 1-forms and they are defined by

$$A(X) = g(X,\rho_1), \qquad B(X) = g(X,\rho_2), \tag{4}$$

and ρ_1, ρ_2 are vector fields associated with 1-forms A, B respectively.

The projective curvature tensor [19] and concircular curvature tensor [20] on a n-dimensional Lorentzian α-Sasakian manifold are respectively defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y], \tag{5}$$

and

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{6}$$

2 Preliminary Notes

A (2n+1)-dimensional differential manifold M is called Lorentzian α-Sasakian manifold if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\phi^2 = I + \eta \circ \xi, \tag{7}$$

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(X, \xi) = \eta(X), \tag{8}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(\phi X, Y) = g(X, \phi Y), \tag{9}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y), \qquad \nabla_X \xi = -\alpha\phi X, \quad \forall\ X, Y \in \Gamma(TM). \tag{10}$$

In a Lorentzian α-Sasakian manifold M, the following relations hold [[23], [21]]:

$$(\nabla_X \phi)Y = \alpha^2\{g(X, Y)\xi - \eta(Y)X\}, \tag{11}$$

$$\eta(R(X, Y)Z) = \alpha^2\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \tag{12}$$

$$R(X, Y)\xi = \alpha^2\{\eta(Y)X - \eta(X)Y\}, \tag{13}$$

$$R(\xi, X)Y = \alpha^2\{g(X, Y)\xi - \eta(Y)X\}, \tag{14}$$

$$S(X, \xi) = (n-1)\alpha^2\eta(X), \tag{15}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\alpha^2\eta(X)\eta(Y). \tag{16}$$

The objective of the present paper is to study Lorentzian α-Sasakian manifold with projective and concircular curvature tensor. After the introduction, in section 3 and 4 we study projectively pseudosymmetric and concircular pseudosymmetric Lorentzian α-Sasakian manifold respectively. In section 5, we study generalized concircular ϕ-recurrent Lorentzian α-Sasakian manifold and proved that generalized concircular ϕ-recurrent Lorentzian α-Sasakian manifold is an Einstein manifold. Finally in the last section we study generalized projective ϕ-recurrent Lorentzian α-Sasakian manifold.

3 Projectively pseudosymmetric Lorentzian α-Sasakian manifold

Definition 3.1 *A Riemannian manifold M is said to be Projectively pseudosymmetric if*

$$R(X, Y) \cdot P = L_P[(X \wedge Y) \cdot P], \tag{17}$$

holds on the set $U_P = \{x \in M : P \neq 0\}$ at x, where L_P is some function on U_P.

Theorem 3.2 *If a n-dimensional Lorentzian α-Sasakian manifold is projectively pseudosymmetric then the manifold is either projectively flat or $L_P = \alpha^2$.*

Proof:

Suppose that Lorentzian α-Sasakian manifold is projectively pseudosymmetric, then we have

$$\begin{aligned} R(X,\xi)P(U,V)\xi - P(R(X,\xi)U,V)\xi - P(U,R(X,\xi)V)\xi \\ -P(U,V)R(X,\xi)\xi = L_P[(X\wedge\xi)P(U,V)\xi - P((X\wedge\xi)U,V)\xi \\ -P(U,(X\wedge\xi)V)\xi - P(U,V)(X\wedge\xi)\xi]. \end{aligned} \tag{18}$$

In view of (1), (14) and (18), one can get

$$\begin{aligned} (L_P-\alpha^2)[\eta(P(U,V)\xi)X - g(X,P(U,V)\xi)\xi - \eta(U)P(X,V)\xi \\ +g(X,U)P(\xi,V)\xi - \eta(V)P(U,X)\xi + g(X,V)P(U,\xi)\xi \\ -\eta(\xi)P(U,V)X + \eta(X)P(U,V)\xi] = 0. \end{aligned} \tag{19}$$

Which implies that either $L_P - \alpha^2 = 0$ or

$$\begin{aligned} \eta(P(U,V)\xi)X - g(X,P(U,V)\xi)\xi - \eta(U)P(X,V)\xi \\ +g(X,U)P(\xi,V)\xi - \eta(V)P(U,X)\xi + g(X,V)P(U,\xi)\xi \\ -\eta(\xi)P(U,V)X + \eta(X)P(U,V)\xi = 0. \end{aligned} \tag{20}$$

If $L_P - \alpha^2 \neq 0$ then we have

$$\begin{aligned} \eta(P(U,V)\xi)X - g(X,P(U,V)\xi)\xi - \eta(U)P(X,V)\xi \\ +g(X,U)P(\xi,V)\xi - \eta(V)P(U,X)\xi + g(X,V)P(U,\xi)\xi \\ -\eta(\xi)P(U,V)X + \eta(X)P(U,V)\xi = 0, \end{aligned} \tag{21}$$

It can be easily verified that in a Lorentzian α-Sasakian manifold, projective curvature tensor satisfies the following conditions:

$$P(X,Y)\xi = 0, \tag{22}$$

$$P(\xi,Y)Z = \alpha^2\{g(Y,Z)\xi - \eta(Z)Y\} - \frac{1}{(n-1)}\{S(Y,Z)\xi \tag{23}$$

$$- \quad (n-1)\alpha^2\eta(Z)Y\}.$$

Using (22) and (24) in (21), we get

$$P(X,Y)X = 0. \tag{24}$$

This completes the proof.

4 Concircularly pseudosymmetric Lorentzian α-Sasakian manifold

Definition 4.1 *A Riemannian manifold is M said to be concircularly pseudosymmetric if*

$$R(X,Y)\cdot C = L_C[(X\wedge Y)\cdot C], \tag{25}$$

holds on the set $U_C = \{x \in M : C \neq 0\}$ *at* x*, where* L_C *is some function on* U_C*.*

Let M be concircularly pseudosymmetric. Then from (25) we get

$$\begin{aligned} R(\xi, Y)C(U, V)W - C(R(\xi, Y)U, V)W - C(U, R(\xi, Y)V)W \\ -C(U, V)R(\xi, Y)W = L_C[(\xi \wedge Y)C(U, V)W - C((\xi \wedge Y)U, V)W \\ -C(U, (\xi \wedge Y)V)W - C(U, V)(\xi \wedge Y)W]. \end{aligned} \tag{26}$$

Using (1) and (14) in (26), it follows that

$$\begin{aligned} (L_C - \alpha^2)[g(C(U, V)W, Y) + \eta(C(U, V)W)\eta(Y) - \eta(U)\eta(C(Y, V)W) \\ +g(Y, U)\eta(C(\xi, V)W) - \eta(V)\eta(C(U, Y)W) + g(Y, V)\eta(C(U, \xi)W) \\ -\eta(W)\eta(C(U, V)Y) + g(Y, W)\eta(C(U, V)\xi)] = 0. \end{aligned} \tag{27}$$

Above equation implies that either $L_C = \alpha^2$ or

$$\begin{aligned} g(C(U, V)W, Y) + \eta(C(U, V)W)\eta(Y) - \eta(U)\eta(C(Y, V)W) \\ +g(Y, U)\eta(C(\xi, V)W) - \eta(V)\eta(C(U, Y)W) + g(Y, V)\eta(C(U, \xi)W) \\ -\eta(W)\eta(C(U, V)Y) + g(Y, W)\eta(C(U, V)\xi) = 0. \end{aligned} \tag{28}$$

Now we proceed the calculation for $L_C \neq \alpha^2$. Making use of equations (6), (12)-(15) in (28), we obtain

$$g(R(U, V)W, Y) = c\{g(V, W)g(Y, U) - g(U, W)g(Y, V)\}, \tag{29}$$

where $c = \alpha^2$. Thus, we can state

Theorem 4.2 *If an* M *be* n*-dimensional Lorentzian* α*-Sasakian manifold* M *is concircularly pseudosymmetric then* M *is either a space of constant curvature or* $L_C = \alpha^2$*.*

5 Generalized Concircular ϕ-recurrent Lorentzian α-Sasakian manifold

Definition 5.1 *A Lorentzian* α*-Sasakian manifold is said to be a generalized concircular* ϕ*-recurrent if its concircular curvature tensor satisfies*

$$\phi^2((\nabla_W C)(X, Y)Z) = A(W)P(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y], \tag{30}$$

where A *and* B *are two 1-forms and they are defined as in (4).*

We consider a Lorentzian α-Sasakian manifold M, which is generalized concircular ϕ-recurrent. Then by virtue of (7), (30) yields

$$(\nabla_W C)(X,Y)Z + \eta((\nabla_W C)(X,Y)Z)\xi = A(W)C(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]. \tag{31}$$

Taking inner product of above with U, we obtain

$$g((\nabla_W C)(X,Y)Z,U) + \eta((\nabla_W C)(X,Y)Z)\eta(U) = A(W)g(C(X,Y)Z,U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \tag{32}$$

Let $\{e_i\}$, $i = 1,2,...,n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (32) and taking summation over i, $1 \leq i \leq n$, we get

$$(\nabla_W S)(X,U) - \frac{dr(W)}{n}g(X,U) + (\nabla_W S)(X,\xi)\eta(U) - \frac{dr(W)}{n}\eta(X)\eta(U) = A(W)\{S(X,U) - \frac{r}{n}g(X,U)\} + (n-1)B(W)g(X,U). \tag{33}$$

We know that

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi). \tag{34}$$

In view of (10) and (15), above equation reduces to

$$(\nabla_W S)(Y,\xi) = \alpha\{(n-1)\alpha^2 g(W,\phi Y) - S(\phi W,Y)\}. \tag{35}$$

Setting $U = \xi$ in (33) and using (8) and (15), it follows that

$$A(W)\{(n-1)\alpha^2 - \frac{r}{n}\}\eta(X) = -(n-1)B(W)\eta(X). \tag{36}$$

Replacing $X = \xi$ in the above equation, one can get

$$A(W)\{(n-1)\alpha^2 - \frac{r}{n}\} = -(n-1)B(W). \tag{37}$$

Thus, we can state:

Theorem 5.2 *In a generalized concircular ϕ-recurrent Lorentzian α-Sasakian manifold, the 1-forms are related by the equation (37)*

Taking contraction in (32) over X and U, we get

$$(\nabla_W S)(Y,Z) - \frac{dr(W)}{n}g(Y,Z) - \frac{dr(W)}{n(n-1)}[-g(Y,Z) - eta(Y)\eta(Z)] = A(W)\{S(Y,Z) - \frac{r}{n}g(Y,Z)\} + (n-1)B(W)g(Y,Z). \tag{38}$$

Putting $Z = \xi$ in (38) and then using (35), we obtain

$$\alpha\{(n-1)\alpha^2 g(W, \phi Y) - S(\phi W, Y)\} = \frac{dr(W)}{n}\eta(Y). \tag{39}$$

Replacing Y by ϕY in (39) and then using (7), (8) and (16), we get

$$S(Y, W)\} = (n-1)\alpha^2 g(Y, W). \tag{40}$$

Hence, we can state the following assertion:

Theorem 5.3 *A generalized concircular ϕ-recurrent Lorentzian α-Sasakian manifold is an Einstein manifold.*

6 Generalized Projective ϕ-recurrent Lorentzian α-Sasakian manifold

Definition 6.1 *A Lorentzian α-Sasakian manifold is said to be generalized projective ϕ-recurrent if its projective curvature tensor satisfies*

$$\phi^2((\nabla_W P)(X, Y)Z) = A(W)C(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y], \tag{41}$$

where A and B are two 1-forms and they are defined as in (4).

By the virtue of (7), equation (41) becomes

$$g((\nabla_W P)(X, Y)Z, U) + \eta((\nabla_W P)(X, Y)Z)\eta(U) = A(W)g(P(X, Y)Z, U) + B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]. \tag{42}$$

Let $\{e_i\}$, $i = 1, 2, ..., n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (42) and taking summation over i, $1 \le i \le n$, we get

$$(\nabla_W S)(X, U) - \frac{1}{(n-1)}\{dr(W)g(X, U) - (\nabla_W S)(X, U)\} + (\nabla_W S)(X, \xi)\eta(U) - \frac{1}{(n-1)}\{dr(W)\eta(X) - (\nabla_W S)(X, \xi)\}\eta(U) = A(W)[S(X, U) - \frac{1}{(n-1)}\{rg(X, U) - S(X, U)\}] + (n-1)B(W)g(X, U). \tag{43}$$

Taking $X = U = \xi$ in (43) and by virtue of (8), (15) and (35), we get from (43) that

$$A(W)\{-n\alpha^2 + \frac{r}{(n-1)}\} = (n-1)B(W). \tag{44}$$

Thus,

Theorem 6.2 *In a generalized projective ϕ-recurrent Lorentzian α-Sasakian manifold, the 1-forms are related by the equation (44)*

References

[1] C.S. Bagewadi and Venkatesha, Some Curvature Tensors on a Trans-Sasakian Manifold, Turk J. Math., 31, 2007, 111-121.

[2] A. Barman, On Lorentzian α-Sasakian manifolds admitting a type of semi-symmetric metric connection, Novi Sad J. Math., 44(2), 2014, 77-88.

[3] D.E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics, 509 Springer-Verlag, Berlin, 1976.

[4] E. Cartan, Surune classe remarquable despaces de Riema, Soc. Math., France, 54, 1926, 214-264.

[5] M.C. Chaki and B. Chaki, On pseudosymmetric manifolds admitting a type of semisymmetric connection, Soochow J. Math., 13, 1987, 1-7.

[6] U.C. De and N. Guha, On generalised recurrent manifolds, Proc. Math. Soc., 7, 1991, 7-11.

[7] R. Deszcz, On pseudosymmetric spaces, Bull. Soc. Math. Belg. Ser. A, 44, 1992, 1-34.

[8] A. Friedmann and J.A. Schouten, Uber die Geometrie der halbsymmetrischen Ubertragung, Math. Z., 21, 1924, 211-223.

[9] A. Gray, L.M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl., 123(4), 1980, 35-58.

[10] H.A. Hayden, Subspaces of a space with torsion, Proc. London Math. Soc., 34, 1932, 27-50.

[11] K. Kenmotsu, A class of almost contact Riemannian manifolds, Thoku Math. J., 24, 1972, 93-103.

[12] J.C. Marrero, The local structure of trans-Sasakian manifolds, Ann. Mat. Pure Appl., 162, 1992, 77-86.

[13] J.A. Oubina, New classes of almost contact metric structures, Publ. Math. Debr., 32(34), 1985, 187-193.

[14] D.G. Prakasha, C.S. Bagewadi and N.S. Basavarajappa, On pseudosymmetric Lorentzian α-Sasakian manifolds, International Journal of Pure and Applied Mathematics, 48(1), 2008, 57-65.

[15] D.G. Prakasha and A. Yildiz, Generalized ϕ-recurrent Lorentzian α-Sasakian manifold, Commun. Fac. Sci. Univ. Ank. Series A1, 59(1), 2010, 53-64.

[16] T. Takahashi, Sasakian ϕ-symmetric spaces, Tohoku Math. J., 29, 1977, 91-113.

[17] S. Tanno, The automorphism groups of almost contact Riemannian manifolds, Tohoku Math. J., 21, 1969, 21-38.

[18] Venkatesha and C.S. Bagewadi, On concircular ϕ-recurrent LP-Sasakian manifolds, Differential Geometry Dynamical Systems, 10, 2008, 312-319.

[19] K. Yano, Concircular geometry I, Concircular transformations, Proc. Imp. Acad. Tokyo, 16, 1940, 195-200.

[20] K. Yano and M. Kon, Structures on Manifolds, Series in Pure Math. World Sci., vol 3, 1984.

[21] A. Yildiz, M. Turan and B.E. Acet, On three-dimensional Lorentzian α-Sasakian manifolds, Bulletin of Mathematical Analysis and Applications, 1(3), 2009, 90-98.

[22] A. Yildiz, M. Turan and C. Murathan, A Class of Lorentzian α-Sasakian manifolds, Kyungpook Math. J., 49, 2009, 789-799.

[23] A. Yildiz and C. Murathan, On Lorentzian α-Sasakian manifolds, Kyungpook Math. J., 45, 2005, 95-103.

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