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Projective change for a new class of (α,β)-metrics

Laurian-Ioan Pişcoran

North University Center of Baia Mare
Technical University of Cluj Napoca
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare, Romania
plaurian@yahoo.com

Lakshmi Narayan Mishra

Department of Mathematics,
Mody University of Science and Technology,
Lakshmangarh, Sikar Road, Sikar, Rajasthan 332 311, India
lakshminarayanmishra04@gmail.com, l_n_mishra@yahoo.co.in

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Abstract

In this paper we will investigate the projective change between recently introduced (α,β)-metric $F=\beta+\frac{a\alpha^2+\beta^2}{\alpha}$ and the (α,β)-metric $\overline{F}=\overline{\alpha}+\overline{\beta}$, where α and $\overline{\alpha}$ are two Riemannian metrics; β and $\overline{\beta}$ are 1-forms and $a\in(0,1]$ is a real positive scalar. Also, we will investigate the locally projectively flatness for the (α,β)-metric F.

1 Introduction

The projective change between two Finsler spaces is well known in literature and have been investigated in a lot of papers (see for example [6],[7],[8],[9],[12]). Recently, in [10], we introduced the new (α,β)-metric

$$F=\beta+\frac{a\alpha^2+\beta^2}{\alpha} \tag{1}$$

where α is a Riemannian metric, β is a 1-form, and $a\in(0,1]$ is a real scalar and we investigate in that paper the projectivelly flatness for this new class of

metrics.
The main purpose of this paper is to continue the study on the above mentioned class of metrics and to investigate the locally projective flatness and also the projective change between this class of (α, β)-metrics and the Randers metrics $\overline{F} = \overline{\alpha} + \overline{\beta}$, where α and $\overline{\alpha}$ are two Riemannian metrics; β and $\overline{\beta}$ are 1-forms and $a \in (0, 1]$ is a real positive scalar.

In Finsler geometry the study of (α, β)-metrics was done in a lot of important papers (see for example [2, 6, 7, 8, 12, 13, 14]) and this type of metrics have importants applications not only in geometry but also in biology ([1]) and other branches of science. We know from [7, 11], that two Finsler metrics F and $\overline{F}$ are projectively related if and only if for their spray coefficients we have the following relation:

$$G^i = \overline{G}^i + P(y)y^i \tag{2}$$

where $P(y)$ is a scalar function on $TM - \{0\}$ and homogeneous of degree one in y.
Also, from [6] we know that a Finsler metric is called a projectively flat metric if it is projectively related to a Minkowskian metric. From [11], we know that the Randers metric $\overline{F} = \overline{\alpha} + \overline{\beta}$ is projectivelly flat if and only if $\overline{\alpha}$ is projectively flat and $\overline{\beta}$ is closed. The projectivelly flatness was investigated in several papers (see [5, 12]).

Definition 1 *([6]) Let*

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right) \tag{3}$$

where G^i are the spray coefficients of F. The tensor $D = D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

Some importants results concerning Douglas metrics are recently presented in [3, 13].

The function $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ and it satisfies the following condition:

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0.$$

Also well known is the fact that F is a Finsler metric if and only if $||\beta_x||_\alpha < b_0$ for any $x \in M$.

In general, the (α, β)-metrics are defined as follows:

Definition 2 *([6]) For a given Riemannian metric* $\alpha = \sqrt{a_{ij}y^iy^j}$ *and one form* $\beta = b_iy^i$*, satisfying* $||\beta_x||_\alpha < b_0$ *for* $\forall x \in M$*, then:* $F = \alpha\phi(s)$*,* $s = \frac{\beta}{\alpha}$*, is called* (α, β)*-metric.*

The covariant derivative of β with respect to α, take the following form: $\nabla\beta = b_{i|j}dx^i \otimes dx^j$. Also, in [6], the following notations are given:

$$r_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right); s_{ij} = \frac{1}{2}\left(b_{i|j} - b_{j|i}\right). \tag{4}$$

It is clear that β is closed if and only if $s_{ij} = 0$. Also, we can take:

$$s_j = b^i s_{ij}; s^i_j = a^{il}s_{lj}; s_0 = s_iy^i; s^i_0 = s^i_jy^i; r_{00} = r_{ij}y^iy^j.$$

If we consider the fundamental tensor of Randers space $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$, then the following formulae are well known from literature:

$$p^i = \frac{1}{\alpha}y^i = a_{ij}\frac{\partial\alpha}{\partial y^j}; \qquad p_i = a_{ij}p^j = \frac{\partial\alpha}{\partial y^i};$$

$$l^i = \frac{1}{L}y^i = g_{ij}\frac{\partial L}{\partial y^j}; \qquad l_i = g^{ij}\frac{\partial L}{\partial y^j} = p_i + b_i;$$

$$l_i = \frac{1}{L}p^i; \quad l^il_j = p^ip_i = 1; \quad l^ip_i = \frac{\alpha}{L};$$

$$p^il_i = \frac{L}{\alpha}; \quad b_ip^i = \frac{\beta}{\alpha}; \quad b_il^i = \frac{\beta}{L}.$$

The geodesic coefficients G^i of F and the geodesic coefficients G^i_α of α, are related as follows (see [6]):

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\left\{\Psi b^i + \Theta\alpha^{-1}y^i\right\} \tag{5}$$

where:

$$Q = \frac{\phi'}{\phi - s\phi'} \tag{6}$$

$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi(\phi - s\phi' + (b^2 - s^2)\phi'')} \tag{7}$$

$$\Psi = \frac{\phi''}{2(\phi - s\phi' + (b^2 - s^2)\phi'')}. \tag{8}$$

In [7] and [11], is presented the condition for an (α, β)-metric to be locally projectively flat, as follows:

Lemma 3 *([7, 11]) A Finsler space* $\Im^n = (M, F)$ *is locally projectivelly flat if and only if:*

$$\frac{\partial F}{\partial x^j} - \frac{\partial^2 F}{\partial x^k \partial y^i}y^k = 0. \tag{9}$$

In [4], is presented the following condition for an (α,β)-metric to be a Douglas metric:

$$\alpha Q\left(s_0^i y^j - s_0^j y^i\right) + \Psi\left(-2\alpha Q s_0 + r_{00}\right)\left(b^i y^j - b^j y^i\right) = \frac{1}{2}\left(G_{kl}^i y^j - G_{kl}^j y^i\right) y^k y^l \tag{10}$$

where $G_{kl}^i = \Gamma_{kl}^i - \gamma_{kl}^i$ and $\gamma_{kl}^i = \frac{\partial^2 G_\alpha^i}{\partial y^k \partial y^l}$.

Theorem 4 *([4]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an (α,β)-metric on an open subset $U \subset R^n (n \geq 3)$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and one form $\beta = b_i y^i \neq 0$. Let $b = ||\beta_x||_\alpha$. Suppose that the following conditions holds:*
(a) β is not parallel with respect to α;
(b) F is not of Randers type;
(c) $db \neq 0$ everywhere or $b =$ constant on U. Then F is a Douglas metric on U, if and only if the function $\phi = \phi(s)$, satisfies the following ODE:

$$\left\{1 + (k_1 + k_2 s^2) s^2 + k_3 s^2\right\} \phi''(s) = (k_1 + k_2 s^2)\left\{\phi(s) - s\phi'(s)\right\} \tag{11}$$

and the covariant derivative $\nabla\beta = b_{i|j} y^i dx^j$ of β with respect to α satisfies the following equation:

$$b_{i|j} = 2\tau\left\{(1 + k_1 b^2) a_{ij} + (k_2 b^2 + k_3) b_i b_j\right\} \tag{12}$$

where $\tau = \tau(x)$ is a scalar function on U and k_1, k_2, k_3 are constants with $(k_2, k_3) \neq (0,0)$.

Remark 5 *The above equation holds good in dimension $n \geq 3$.*

Finally, let's remark that some interesting results can be obtained using (α,β)-metrics in Finsler-Lagrange geometry (see [16]). Also, in Lagrange-Finsler geometry an important problem is that concerning the Whitney extension theorem. In [15] are presented some results concerning the Whitney extension theorem in differential geometry.

2 Main Results

Now, using the above mentioned Theorem 4, we will compute for the (α,β)-metric (1), the coefficients $b_{i|j}$, taking into account that $F = \alpha\phi(s)$, where $\phi(s) = s^2 + s + a$, where $a \in (0,1]$ is a positive scalar.
After tedious computations, we get:

$$b_{i|j} = \tau\left[\left(1 + \frac{2}{a} b^2\right) a_{ij} - \frac{3}{a} b_i b_j\right]. \tag{13}$$

Next, we compute:

$$r_{00}=\tau\left[\left(1+\frac{2}{a}b^2\right)\alpha^2-\frac{3}{a}\beta^2\right]. \tag{14}$$

Using (6), (7), (8), for $\phi(s)=s^2+s+a$, we get:

$$Q=\frac{\phi'}{\phi-s\phi'}=\frac{2s+1}{a-s^2} \tag{15}$$

$$\Theta=\frac{\phi\phi'-s(\phi\phi''+\phi'\phi')}{2\phi(\phi-s\phi'+(b^2-s^2)\phi'')}=\frac{a-s^2(4s+3)}{2(s^2+s+a)(2b^2-3s^2+a)} \tag{16}$$

$$\Psi=\frac{\phi''}{2(\phi-s\phi'+(b^2-s^2)\phi'')}=\frac{1}{2b^2-3s^2+a}. \tag{17}$$

Substituing (15), (16), (17) in (5), we get:

$$G^i=G^i_\alpha+\frac{\alpha^2(2\beta+\alpha)}{a\alpha^2-\beta^2}s^i_0$$

$$+\left\{\frac{-2\alpha(2\beta+\alpha)}{a\alpha^2-\beta^2}s_0+r_{00}\right\}\left\{\frac{\alpha^2b^i}{2b^2\alpha^2-3\beta^2+a\alpha^2}+\frac{(a\alpha^3-4\beta^3-3\alpha\beta^2)y^i}{(2\beta^2+2\alpha\beta+2a\alpha^2)(2b^2\alpha^2-3\beta^2+a\alpha^2)}\right\} \tag{18}$$

where r_{00} is given in (14).

Remark 6 *Now, we can formulate the first result: the the (α,β)-metric (1), is a Douglas metric with respect to Theorem 4, if and only if (13) take place:*

$$b_{i|j}=\tau\left[\left(1+\frac{2}{a}b^2\right)a_{ij}-\frac{3}{a}b_ib_j\right]$$

for some scalar function $\tau=\tau(x)$, where $b_{i|j}$ represents the coefficients of the covariant derivative $\beta=b_iy^i$ with respect to α.In this case β is closed.

If β is closed, then one obtains: $s_{ij}=0\Rightarrow b_{i|j}=b_{j|i}$ and $s^i_0=0; s_0=0$. Replacing (14) in (18), we get:

$$G^i=G^i_\alpha-\tau\left[\frac{-a\alpha^3+4\beta^3+3\alpha\beta^2}{a(2\beta^2+2\alpha\beta+2a\alpha^2)}\right]y^i+\tau\frac{\alpha^2}{a}b^i. \tag{19}$$

We consider a scalar function $P=P(y)$ on $TM-\{0\}$, i.e.:

$$G^i=G^i_{\overline{\alpha}}+Py^i. \tag{20}$$

From (19) and (20), we get:

$$P+\tau\left[\frac{-a\alpha^3+4\beta^3+3\alpha\beta^2}{a(2\beta^2+2\alpha\beta+2a\alpha^2)}\right]y^i=G^i_\alpha-G^i_{\overline{\alpha}}+\tau\frac{\alpha^2}{a}b^i. \tag{21}$$

Using the fact that in the previous equation (21), the RHS of this equation is a quadratic form, thus there must be a one form $\theta = \theta_i y^i$, such that:

$$P + \tau \left[\frac{-a\alpha^3 + 4\beta^3 + 3\alpha\beta^2}{a(2\beta^2 + 2\alpha\beta + 2a\alpha^2)} \right] = \theta$$

Then, can obtain:

$$G^i = G^i_{\overline{\alpha}} + \theta y^i - \tau \frac{\alpha^2}{a} b^i. \tag{22}$$

Now using (13) and (22) and also the above results, we can formulate the following theorem:

Theorem 7 *Let $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ and $\overline{F} = \overline{\alpha} + \overline{\beta}$, be two (α, β)-metrics, where α and $\overline{\alpha}$ are two Riemannian metrics; β and $\overline{\beta}$ are 1-forms and $a \in (0, 1]$ is a real positive scalar. Then F is projectively related to $\overline{F}$, if and only if the followings equations, holds:*

$$G^i = G^i_{\overline{\alpha}} + \theta y^i - \tau \frac{\alpha^2}{a} b^i$$

$$b_{i|j} = \tau \frac{1}{a} \left[\left(a + 2b^2 \right) a_{ij} - 3b_i b_j \right]$$

$$d\overline{\beta} = 0$$

where $b^i = a^{ij} b_j$; $b = ||\beta||_\alpha$ and $b_{i|j}$ are the coefficients of the covariant derivatives of β with respect to α; $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M.

The proof is immediate using the already proved relations (13) and (22). Also, we can now formulate the following corollary:

Corollary 8 *Let $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ and $\overline{F} = \overline{\alpha} + \overline{\beta}$, be two (α, β)-metrics on a manifold M, where α and $\overline{\alpha}$ are two Riemannian metrics; β and $\overline{\beta}$ are 1-forms and $a \in (0, 1]$ is a real positive scalar. Then F is projectively flat if the following relation holds:*

$$G^i = G^i_{\overline{\alpha}} + \theta y^i - \tau \frac{\alpha^2}{a} b^i$$

where $b^i = a^{ij} b_j$; $b = ||\beta||_\alpha$ and $b_{i|j}$ are the coefficients of the covariant derivatives of β with respect to α; $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M.

The proof is imediate using the above Theorem 7.

Theorem 9 *Let $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ the (α,β)-metric introduced in (1) on an n-dimensional manifold M, with α a Riemannian metric; β a 1-form and $a \in (0,1]$ is a real positive scalar. Then F is locally projectively flat if and only if:*

$$2\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left[\frac{\partial\beta}{\partial x^k} - \frac{\beta}{\alpha}\frac{\partial\alpha}{\partial x^k}\right]y^k +$$
$$\left(1+\frac{2\beta}{\alpha}\right)\left(\frac{\partial b_i}{\partial x^k} - \frac{\partial b_k}{\partial x^i}\right)y^k + \left(a - \frac{\beta^2}{\alpha^2}\right)\left[\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k - \frac{\partial\alpha}{\partial x^i}\right] = 0. \quad (23)$$

We apply Lemma 3, using

$$\frac{\partial F}{\partial x^j} - \frac{\partial^2 F}{\partial x^k \partial y^i}y^k = 0.$$

First, we compute:

$$\frac{\partial F}{\partial x^k} = \frac{\partial}{\partial x^k}\left(\beta + a\alpha + \frac{\beta^2}{\alpha}\right) = \frac{\partial\beta}{\partial x^k} + a\frac{\partial\alpha}{\partial x^k} + \frac{2\beta}{\alpha}\frac{\partial\beta}{\partial x^k} - \frac{\beta^2}{\alpha^2}\frac{\partial\alpha}{\partial x^k} =$$
$$\left(1+\frac{2\beta}{\alpha}\right)\frac{\partial\beta}{\partial x^k} + \left(a - \frac{\beta^2}{\alpha^2}\right)\frac{\partial\alpha}{\partial x^k}. \quad (24)$$

Then, we compute:

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k = 2\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\frac{\partial\beta}{\partial x^k}y^k + \left(1+\frac{2\beta}{\alpha^2}\right)\frac{\partial}{\partial y^i}\left(\frac{\partial\beta}{\partial x^k}\right)y^k - 2y^k\left(\frac{\beta}{\alpha}\right)\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\frac{\partial\alpha}{\partial x^k} +$$
$$\left(a - \frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k =$$
$$2\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha^2}\right)\frac{\partial\beta}{\partial x^k}y^k + \left(1+\frac{2\beta}{\alpha}\right)\frac{\partial b^i}{\partial x^k}y^k - 2\left(\frac{\beta}{\alpha}\right)\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\frac{\partial\alpha}{\partial x^k}y^k +$$
$$\left(a - \frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k. \quad (25)$$

From (24), replacing k and i and substituing $\beta = b_k(x)y^k$, one obtains:

$$\frac{\partial F}{\partial x^i} = \left(1+\frac{2\beta}{\alpha}\right)\frac{\partial b_k}{\partial x^i}y^k + \left(a - \frac{\beta^2}{\alpha^2}\right)\frac{\partial\alpha}{\partial x^i}. \quad (26)$$

Finally, replacing (24); (25) and (26) in (9), we get:

$$2\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\frac{\partial\beta}{\partial x^k}y^k + \left(1+\frac{2\beta}{\alpha}\right)\frac{\partial b_i}{\partial x^k}y^k - 2\left(\frac{\beta}{\alpha}\right)\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\frac{\partial\alpha}{\partial x^k}y^k + \left(a - \frac{\beta^2}{\alpha^2}\right)\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k -$$
$$\left(1+\frac{2\beta}{\alpha}\right)\frac{\partial b_k}{\partial x^i}y^k - \left(a - \frac{\beta^2}{\alpha^2}\right)\frac{\partial\alpha}{\partial x^i} = 0 \quad (27)$$

and from this, we get the asertion of the theorem. The converse follow also easily.

Theorem 10 *Let $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ the (α,β)-metric given by (1), be locally projectively flat. Asume that α is locally projectively flat. Then:*

$$\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)(2P-Q) = \frac{1}{2}\left(\frac{1}{2\beta}+\frac{1}{\alpha}\right)\left(\frac{\partial b_k}{\partial x^i}-\frac{\partial b_i}{\partial x^k}\right)y^k \tag{28}$$

where $P = \frac{1}{2\alpha}\frac{\partial\alpha}{\partial x^k}y^k$; $Q = \frac{1}{2\beta}\frac{\partial\beta}{\partial x^k}y^k$.

Because α is locally projectively flat, from (9), we get:

$$\frac{\partial}{\partial y^i}\left(\frac{\partial\alpha}{\partial x^k}\right)y^k - \frac{\partial\alpha}{\partial x^i} = 0. \tag{29}$$

From (23) and (29), we get:

$$2\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left[\frac{\partial\beta}{\partial x^k}-\frac{\beta}{\alpha}\frac{\partial\alpha}{\partial x^k}\right]y^k = \left(1+\frac{2\beta}{\alpha}\right)\left[\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i}\right]y^k. \tag{30}$$

Using the above definitions of P and Q and dividing with 2β in (30), we get:

$$\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)(2P-Q) = \frac{1}{2}\left(\frac{1}{2\beta}+\frac{1}{\alpha}\right)\left(\frac{\partial b_k}{\partial x^i}-\frac{\partial b_i}{\partial x^k}\right)y^k$$

and this complete the proof.

Recently, in [14], was introduced the following:

Definition 11 *([14]) We say that an (α,β)-metric $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ on a manifold M, satisfy the sign property, if the function*

$$A_\phi(s) := \phi'(-s)\phi(s) + \phi(-s)\phi'(s)$$

has a fix sign on a symmetric interval $(-b_0, b_0)$. Here, with s is denoted $s = \frac{\beta}{\alpha}$.

Using this important definition, we will try now to investigate in which condition, the metric (1) satisfy this new property. So, we give the following:

Example 12 *Let us consider the metric (1), $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$, with $\phi(s) = s^2 + s + a$, where $a \in (0, 1]$ is a positive scalar.*
In this case, we have:

$$A_\phi(s) := \phi'(-s)\phi(s) + \phi(-s)\phi'(s) = -2s^2 + 2a.$$

We conclude that, for $s \in (-\sqrt{a}, \sqrt{a})$, $A_\phi(s)$ has a fix sign. Thus this metric satisfy the sign property.

Remark 13 *The above quantities P and Q from Theorem 10, were presented in [11] and used in that paper to describe the locally projectively flatness of some other important (α,β)-metrics, such as: Kropina and Matsumoto metrics.*

3 Conclusion

In this paper we have continued the investigations started on paper [10] on the new family of (α, β)-metrics $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ and we succed to obtain some important results concerning the projective change and locally projective flatness of this type of metrics. Also the sign property for this new type of (α, β)-metrics were investigated. In our future papers we will try to investigate other important properties for this class of (α, β)-metrics.

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