Mathematica Aeterna, Vol. 6, 2016, no. 6, 895 - 906

Determinant Systems Method for Computing Reflexive Generalized Inverses of Products of Fredholm Operators

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Abstract

In the paper, for given Fredholm operators $A \in L(Y, Z)$ and $B \in L(X, Y)$, an explicit construction for reflexive generalized inverses of A and B such that their product in reverse order is a reflexive generalized inverse of AB, is provided. In this approach, based on results of the theory of determinant systems, the correspondence between considered Fredholm operators and their determinant systems is applied.

Mathematics Subject Classification: 47A53, 47A05, 15A09

Keywords: Fredholm operator, determinant system, reflexive generalized inverse, reverse order law

1 Introduction

Determinants for operators acting on infinite dimensional linear spaces provide important tools for solving linear equations. As in the finite dimensional case, by means of determinants, we are able to exhibit explicit solutions of the equations induced by the considered operators. Formulas for the solutions are generalizations of the Cramer's rule. Given a determinant system for a Fredholm operator A, we solve the equation $Ax = y_0$, where y_0 belongs to the range of A. By employing terms of the determinant system, we obtain a reflexive generalized inverse G of A and therefore we arrive at the solution $x = Gy_0 + \tilde{x}$, $\tilde{x}$ being any element of the null space of A.

In the fifties of the twentieth century there were created two independent theories of determinants in Banach spaces. The first one was proposed by A.F. Ruston [29] and developed by A. Grothendieck [18]. The second one, the theory of determinant systems to which we refer, was created by T. Leżański [20], modified by R. Sikorski [4, 34, 35, 36] and completed by A. Buraczewski [2, 3, 5]. Our primary concern in this paper is the problem of the relationship between reflexive generalized inverses of the product AB of two Fredholm operators on linear spaces and the product of reflexive generalized inverses of B and A. We combine results of the theory of determinant systems with the results of the theory of generalized inverses.

Let $A \in L(X,Y)$, X, Y being linear spaces, and consider the following equations:

$$(1)\, AGA = A, \quad (2)\, GAG = G, \quad (3)\, (AG)^* = AG, \quad (4)\, (GA)^* = GA\,, \qquad (1)$$

where C^* stands for the adjoint of C. Any $G \in L(Y,X)$ satisfying some of the above equations is said to be a *generalized inverse of* A. Given $\emptyset \neq \eta \subseteq \{1,2,3,4\}$, $A\eta$ denotes the set of all operators G which satisfy (i) of (1) for all $i \in \eta$. Any $G \in A\eta$ is called an η-inverse of A. The unique $\{1,2,3,4\}$-inverse of A is said to be the *Moore-Penrose inverse of* A and is denoted by $A^\dagger$ [24, 26]. Theory and computations of generalized inverses of linear operators are important subjects in many branches of applied science, such as matrix analysis, statistics, engineering and numerical linear algebra [1, 6, 7, 17, 28]. So far, thousands of papers on various aspects of generalized inverses and their applications have appeared. We mentioned here only a small part of the still growing bibliography on generalized inverses.

Analogues of the reverse order law $(AB)^{-1} = B^{-1}A^{-1}$ for invertible operators have been examined intensively for various types of generalized inverses. First T. N. E. Greville [16] proved that for matrices $A \in C^{n\times m}$ and $B \in C^{m\times n}$,

$$(AB)^\dagger = B^\dagger A^\dagger \qquad (2)$$

if and only if $R(A^*AB) \subseteq R(B)$ and $R(BB^*A^*) \subseteq R(A^*)$. Since then, many equivalent conditions for (2) have been discovered. Most books and articles have been concerned with the matrix case [1, 14, 21, 23, 27, 28, 31, 32, 33, 38, 42]. In a lot of papers results related to generalized inverses of matrices were extended to generalized inverses of linear operators in infinite dimensional linear spaces. In this case, usually additional topological structures, like Banach or Hilbert spaces, were assumed [15, 25]. Different classes of generalized inverses, such as $\{1,2,3\}$–inverses, $\{1,2,4\}$–inverses, $\{1,3\}$–inverses and $\{1,4\}$–inverses on the sets of both matrices and bounded linear operators, were also considered in [12, 22]. In [19] results from [41], derived for matrices, were extended to $\{1,2\}$-inverses of operators acting between arbitrary linear spaces.

In the matrix case the reverse order law for $\{1,2\}$-inverses of the product of two matrices was studied in [14], where the authors obtained some results using the product singular value decomposition of two matrices. The authors of [13] improved the results from [14] for the case of regular bounded linear operators on Hilbert spaces.

In the literature, generalized inverses belonging to $A\{1,2\}$, studied in this paper, are called $\{1,2\}$–inverses of A [1] or algebraic generalized inverses of A [25]. Other sources refer to $\{1,2\}$–inverses as reflexive generalized inverses [13, 40] or quasi-inverses [4, 37, 39]. We address the problem of when the product of reflexive generalized inverses of two Fredholm operators A and B, in reverse order, is again a reflexive generalized inverse of the product AB. We rely on a description of a reflexive generalized inverse of a Fredholm operator using terms of a determinant system for this operator [5, 8, 10]. In the derivation of the main result some ideas, developed in [3] for Fredholm endomorphisms, are used. We also utilize the result from [19] and give a direct, constructive solution to the problem mentioned above. The method proposed in the paper yields purely algebraic conditions and it can be applied in more general settings than that of matrices.

2 Preliminaries

We begin by recalling the main notions and facts concerning the theory of determinant systems needed for our purpose. The notation is adopted from papers [2, 4, 8, 9, 10, 11, 36].

A pair (Ξ, X), where Ξ, X are linear spaces over the same real or complex field F, is said to be a *pair of conjugate linear spaces*, if there exists a bilinear functional $I : \Xi \times X \to F$, whose value at a point $(\xi, x) \in \Xi \times X$ is denoted by ξx and which fulfills the conditions:
(a1) $\xi x = 0$ for every $x \in X$ implies $\xi = 0$;
(a2) $\xi x = 0$ for every $\xi \in \Xi$ implies $x = 0$.
The functional I is called the *scalar product on* $\Xi \times X$.

In what follows, (Ξ, X), (Ω, Y) and (Λ, Z) denote pairs of conjugate linear spaces (over F) with scalar products I, J and K on $\Xi \times X$, $\Omega \times Y$ and $\Lambda \times Z$, respectively. A bilinear functional $A : \Omega \times X \to F$, ωAx being its value at (ω, x), simultaneously interpreted as a linear operator $\xi = \omega A$ acting from Ω into Ξ and as a linear operator $y = Ax$ acting from X into Y, defined by the relationship $\omega Ax = (\omega A)x = \omega(Ax)$, is said to be a (Ξ, Y)-*operator on* $\Omega \times X$. Let $op(\Omega \to \Xi, X \to Y)$ denote the linear space of all (Ξ, Y)-operators on $\Omega \times X$. Clearly, any $A \in op(\Omega \to \Xi, X \to Y)$, interpreted as $A : \Omega \to \Xi$, is the adjoint of $A : X \to Y$. We introduce the following notation for ranges and null spaces of the operators:

$$R(A)=\{Ax : x\in X\}, \qquad N(A)=\{x\in X : Ax=0\},$$

$$\mathcal{R}(A)=\{\omega A : \omega\in \Omega\}, \qquad \mathcal{N}(A)=\{\omega\in\Omega : \omega A=0\}.$$

For fixed nonzero elements $y_0\in Y$, $\xi_0\in\Xi$, the symbol $y_0\cdot\xi_0$ stands for the operator defined by the formula

$$\omega(y_0\cdot\xi_0)x=(\omega y_0)(\xi_0 x) \ \text{ for } \ (\omega,x)\in\Omega\times X\,.$$

Any finite sum of these operators is called a *finitely dimensional operator on* $\Omega\times X$. An operator $A\in op(\Omega\to\Xi, X\to Y)$ such that $\dim N(A)=n<\infty$, $\dim\mathcal{N}(A)=m<\infty$, $R(A)=\mathcal{N}(A)^{\perp}$ and $\mathcal{R}(A)=N(A)^{\perp}$ is said to be a *Fredholm operator on* $\Omega\times X$ *of order* $r(A)=\min\{n,m\}$ *and index* $d(A)=n-m$.

If D is a $(\mu+m)$–linear functional on $\Xi^{\mu}\times Y^{m}$, then $D\begin{pmatrix}\xi_1,\dots,\xi_\mu\\ y_1,\dots,y_m\end{pmatrix}$ denotes its value at a point $(\xi_1,\dots,\xi_\mu,y_1,\dots,y_m)\in\Xi^{\mu}\times Y^{m}$. D is called *bi-skew symmetric* if it is skew symmetric both in variables $\xi_1,\dots,\xi_\mu$ and $y_1,\dots,y_m$. The set of all bi-skew symmetric functionals on $\Xi^{\mu}\times Y^{m}$ is denoted by $bss_{\mu,m}(\Xi,Y)$. D is said to be an (Ω,X)*– functional on* $\Xi^{\mu}\times Y^{m}$ if it satisfies the conditions:

(b1) for arbitrary fixed elements $\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_\mu\in\Xi$ and $y_1,\dots,y_m\in Y$ there exists an element $x_i\in X$ such that

$$\xi x_i=D\begin{pmatrix}\xi_1, & \dots, & \xi_{i-1},\xi,\xi_{i+1}, & \dots, & \xi_\mu\\ y_1, & & \dots, & & y_m\end{pmatrix}$$

for every $\xi\in\Xi$ and $i=1,\dots\mu$;

(b2) for arbitrary fixed elements $\xi_1,\dots,\xi_\mu\in\Xi$ and $y_1,\dots,y_{j-1},y_{j+1},\dots,y_m\in Y$ there exists an element $\omega_j\in\Omega$ such that

$$\omega_j y=D\begin{pmatrix}\xi_1, & & \dots, & & \xi_\mu\\ y_1, & \dots, & y_{j-1},y,y_{j+1}, & \dots, & y_m\end{pmatrix}$$

for every $y\in Y$ and $j=1,\dots m$. We use $L_{\mu,m}(\Xi,Y)$ to denote the set of all (Ω,X)– functionals on $\Xi^{\mu}\times Y^{m}$.

A sequence $(D_n)_{n\in N_0}$ is called a *determinant system for* $A\in op(\Omega\to\Xi,X\to Y)$, if

(c1) $D_n\in bss_{\mu_n,m_n}(\Xi,Y)$, where $\mu_n,m_n\in N_0$, $\mu_n=\mu_0+n,\ m_n=m_0+n$ and $\min\{\mu_0,m_0\}=0$;

(c2) $D_n\in L_{\mu_n,m_n}(\Xi,Y)$;

(c3) there exists $r\in N_0$ such that $D_r\neq 0$;

(c4) the following identities hold for $n\in N_0$:

$$D_{n+1}\begin{pmatrix} \xi_0, & \dots, & \xi_{\mu_n} \\ Ax, y_1, & \dots, & y_{m_n} \end{pmatrix} = \sum_{i=0}^{\mu_n} (-1)^i \xi_i x\, D_n \begin{pmatrix} \xi_0, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_{\mu_n} \\ y_1, \dots, y_{m_n} \end{pmatrix},$$

$$D_{n+1}\begin{pmatrix} \omega A, \xi_1, & \dots, & \xi_{\mu_n} \\ y_0, & \dots, & y_{m_n} \end{pmatrix} = \sum_{j=0}^{m_n} (-1)^j \omega y_j\, D_n \begin{pmatrix} \xi_1, \dots, \xi_{\mu_n} \\ y_0, \dots, y_{j-1}, y_{j+1}, \dots, y_{m_n} \end{pmatrix},$$

where $x \in X, \omega \in \Omega,\ \xi_i \in \Xi, y_j \in Y,\ i = 0, 1, \dots, \mu_n,\ j = 0, 1, \dots, m_n$. The least $r \in N_0$ such that D_r does not vanish identically is said to be the *order of* $(D_n)_{n\in N_0}$ and is denoted by $r\left((D_n)_{n\in N_0}\right)$. The integer $d\left((D_n)_{n\in N_0}\right) = \mu_0 - m_0$ is called the *index of* $(D_n)_{n\in N_0}$.

It is well known [2, 36], that A has a determinant system if and only if A is Fredholm. Moreover, $r\left((D_n)_{n\in N_0}\right) = r(A)$ and $d\left((D_n)_{n\in N_0}\right) = d(A)$, $(D_n)_{n\in N_0}$ being any determinant system for A.

3 Main Results

In this section we describe reflexive generalized inverses of products of Fredholm operators acting between arbitrary linear spaces. As a tool of the description we use terms of determinant systems for the respective operators.

Throughout the paper, $A_1 \in op(\Omega \to \Xi, X \to Y)$, $A_2 \in op(\Lambda \to \Omega, Y \to Z)$ always denote Fredholm operators of orders $r(A_1) = r' = \min\{n', m'\}$, $r(A_2) = r'' = \min\{n'', m''\}$ and indices $d(A_1) = d' = n' - m'$, $d(A_2) = d'' = n'' - m''$, respectively. By definitions of A_1, A_2, there exist subspaces $Y' \subset Y$, $\Omega'' \subset \Omega$ such that $\dim Y' = m'$, $\dim \Omega'' = n''$ and the following direct sum decompositions hold:

$$Y = R(A_1) \oplus Y', \qquad \Omega = \mathcal{R}(A_2) \oplus \Omega''. \tag{3}$$

We begin with the following auxiliary result.

Lemma 3.1. *Suppose that:*

(a) $B_1 \in op(\Xi \to \Omega, Y \to X)$ *and* $B_2 \in op(\Omega \to \Lambda, Z \to Y)$ *are reflexive generalized inverses of* A_1 *and* A_2 *, respectively;*

(b) $(x'_1, \dots, x'_{n'})$, $(\omega'_1, \dots, \omega'_{m'})$, $(y''_1, \dots, y''_{n''})$ *and* $(\lambda''_1, \dots, \lambda''_{m''})$ *are bases of* $N(A_1)$, $\mathcal{N}(A_1)$, $N(A_2)$ *and* $\mathcal{N}(A_2)$, *respectively;*

(c) $(y''_1, \dots, y''_{\tilde{n}''})$ *and* $(\omega'_1, \dots, \omega'_{\tilde{m}'})$ *are bases of* $Y_1 = N(A_2) \cap R(A_1)$ *and* $\Omega_1 = \mathcal{N}(A_1) \cap \mathcal{R}(A_2)$, *g respectively.*

Then

$$N(A_2A_1) = \operatorname{span}(x'_1, \dots, x'_{n'},\ B_1y''_1, \dots, B_1y''_{\tilde{n}''}) \tag{4}$$

and

$$\mathcal{N}(A_2A_1) = \operatorname{span}(\lambda''_1, \dots, \lambda''_{m''},\ \omega'_1B_2, \dots, \omega'_{\tilde{m}'}B_2), \tag{5}$$

$(x'_1, \dots, x'_{n'},\ B_1y''_1, \dots, B_1y''_{\tilde{n}''})$, $(\lambda''_1, \dots, \lambda''_{m''},\ \omega'_1B_2, \dots, \omega'_{\tilde{m}'}B_2)$ *being linearly independent.*

Proof. In view of the relationship between Fredholm operators and their reflexive generalized inverses [2], there exist elements $y'_1, \dots, y'_{m'} \in Y$, $\xi'_1, \dots, \xi'_{n'} \in \Xi$, $\omega''_1, \dots, \omega''_{n''} \in \Omega$, $z''_1, \dots, z''_{m''} \in Z$ such that $\xi'_i x'_j = \delta_{ij}$ $(i, j = 1, \dots, n')$, $\omega'_i y'_j = \delta_{ij}$ $(i, j = 1, \dots, m')$, $\omega''_i y''_j = \delta_{ij}$ $(i, j = 1, \dots, n'')$, $\lambda''_i z''_j = \delta_{ij}$ $(i, j = 1, \dots, m'')$ and the following identities are satisfied:

$$B_1 A_1 = I - \sum_{i=1}^{n'} x'_i \cdot \xi'_i, \quad A_1 B_1 = J - \sum_{i=1}^{m'} y'_i \cdot \omega'_i, \tag{6}$$

$$B_2 A_2 = J - \sum_{i=1}^{n''} y''_i \cdot \omega''_i, \quad A_2 B_2 = K - \sum_{i=1}^{m''} z''_i \cdot \lambda''_i\,. \tag{7}$$

It is obvious that $A_2 A_1 x'_i = A_2(A_1 x'_i) = 0$ for $i = 1, \dots, n'$. Since $y''_j \in \mathcal{N}(A_1)^\perp$ for any $j = 1, \dots, \tilde{n}''$, we obtain

$$A_2 A_1 (B_1 y''_j) = A_2 \left(J - \sum_{i=1}^{m'} y'_i \cdot \omega'_i \right) y''_j = A_2 y''_j - \sum_{i=1}^{m'} A_2 y'_i \cdot \omega'_i y''_j = 0\,.$$

Thus, elements

$$x'_1, \dots, x'_{n'}, B_1 y''_1, \dots, B_1 y''_{\tilde{n}''} \tag{8}$$

are solutions of the equation $A_2 A_1 x = 0$. Let $\sum_{i=1}^{n'} \alpha_i x'_i + \sum_{i=1}^{\tilde{n}''} \beta_i B_1 y''_i = 0$. Therefore, by $\sum_{i=1}^{n'} \alpha_i A_1 x'_i + \sum_{i=1}^{\tilde{n}''} \beta_i A_1 B_1 y''_i = 0$, we get $\sum_{i=1}^{\tilde{n}''} \beta_i A_1 B_1 y''_i = 0$. Taking the value of $\sum_{i=1}^{\tilde{n}''} \beta_i A_1 B_1 y''_i$ at ω''_j $(j = 1, \dots, \tilde{n}'')$ and bearing in mind (6), we find

$$\sum_{i=1}^{\tilde{n}''} \beta_i \omega''_j A_1 B_1 y''_i = \sum_{i=1}^{\tilde{n}''} \beta_i \left(\omega''_j y''_i - \sum_{k=1}^{m'} \omega''_j y'_k \cdot \omega'_k y''_i \right) = \sum_{i=1}^{\tilde{n}''} \beta_i \delta_{ji} = 0\,,$$

which implies $\beta_j = 0$ for $j = 1, \dots, \tilde{n}''$. Furthermore, it follows from the linear independence of $x'_1, \dots, x'_{n'}$, that $\alpha_i = 0$ $(i = 1, \dots, n')$. Consequently, the elements (8) are linearly independent. Given $x \in N(A_2 A_1)$, we obtain $B_1 B_2 A_2 A_1 x = 0$. Next, by (7), we get $B_1 A_1 x - \sum_{i=1}^{n''} B_1 y''_i \cdot \omega''_i A_1 x = 0$. Taking into account (6), we arrive at $x = \sum_{i=1}^{n'} \xi'_i x \cdot x'_i + \sum_{i=1}^{n''} B_1 y''_i \cdot \omega''_i A_1 x$. Remembering that $y''_i \notin R(A_1)$ for $i = \tilde{n}'' + 1, \dots, n''$, we find $B_1 y''_i = 0$ and conclude that each solution of $A_2 A_1 x = 0$ is a linear combination of elements (8). This proves (4).

For any $i = 1, \ldots, m''$ we have $\lambda''_i A_2 A_1 = (\lambda''_i A_2) A_1 = 0$. Since $\omega'_j \in N(A_2)^\perp$ for $j = 1, \ldots, \tilde{m}'$, in view of (7), we obtain

$$\omega'_j B_2 A_2 A_1 = \omega'_j A_1 - \sum_{i=1}^{n''} \omega'_j y''_i \cdot \omega''_i A_1 = 0$$

and deduce that the elements

$$\lambda''_1, \ldots, \lambda''_{m''}, \omega'_1 B_2, \ldots, \omega'_{\tilde{m}'} B_2 \tag{9}$$

are solutions of the equation $\lambda A_2 A_1 = 0$. Assume $\sum\limits_{i=1}^{m''} \alpha_i \lambda''_i + \sum\limits_{i=1}^{\tilde{m}'} \beta_i \omega'_i B_2 = 0$. Then the identity $\sum\limits_{i=1}^{m''} \alpha_i \lambda''_i A_2 + \sum\limits_{i=1}^{\tilde{m}'} \beta_i \omega'_i B_2 A_2 = 0$ yields $\sum\limits_{i=1}^{\tilde{m}'} \beta_i \omega'_i B_2 A_2 = 0$. By (7),

$$\sum_{i=1}^{\tilde{m}'} \beta_i \omega'_i B_2 A_2 y'_j = \sum_{i=1}^{\tilde{m}'} \beta_i \left(\omega'_i y'_j - \sum_{k=1}^{n''} \omega'_i y''_k \cdot \omega''_k y'_j \right) = \sum_{i=1}^{\tilde{m}'} \beta_i \delta_{ij} = 0.$$

Consequently, $\beta_j = 0$ $(j = 1, \ldots, \tilde{m}')$. Moreover, it follows from the linear independence of $\lambda''_1, \ldots, \lambda''_{m''}$ that $\alpha_i = 0$ $(i = 1, \ldots, m'')$, which imply that (9) are also linearly independent. Given $\lambda \in \mathcal{N}(A_2 A_1)$, we have $\lambda A_2 A_1 B_1 B_2 = 0$. Thus, by (6), $\lambda A_2 B_2 - \sum\limits_{i=1}^{m'} \lambda A_2 y'_i \cdot \omega'_i B_2 = 0$. Bearing in mind (7), we get $\lambda = \sum\limits_{i=1}^{m''} \lambda z''_i \cdot \lambda''_i - \sum\limits_{i=1}^{m'} \lambda A_2 y'_i \cdot \omega'_i B_2$. Since $\omega'_i \notin \mathcal{R}(A_2)$ for any $i = \tilde{m}' + 1, \ldots, m'$, we obtain $\omega'_i B_2 = 0$. Therefore, each solution of the homogeneous equation $\lambda A_2 A_1 = 0$ is a linear combination of elements (9). This shows that (5) is valid. □

The next theorem yields a sufficient condition for the product $B_1 B_2$ of reflexive generalized inverses B_1, B_2 of Fredholm operators A_1, A_2, respectively, to be a reflexive generalized inverse of the product $A_2 A_1$.

By (3.1), $d(A_2 A_1) = n' + \tilde{n}'' - (m'' + \tilde{m}')$. Moreover, bearing in mind the classical index theorem [30], $d(A_2 A_1) = d' + d'' = n' - m' + n'' - m''$. Thus, $n'' - \tilde{n}'' = m' - \tilde{m}'$. Denote $t = n'' - \tilde{n}''$.

$$N(A_2) = Y_1 \oplus Y_2 , \qquad \mathcal{N}(A_1) = \Omega_1 \oplus \Omega_2 , \tag{10}$$

where $Y_2 = N(A_2) \cap Y'$, $\Omega_2 = \mathcal{N}(A_1) \cap \Omega''$ and $\dim Y_2 = \dim \Omega_2 = t$. Assume $Y_2 = \operatorname{span}(y_1, \ldots, y_t)$, $\Omega_2 = \operatorname{span}(\omega_1, \ldots, \omega_t)$ and $\omega_i y_j = \delta_{ij}$ $(i, j = 1, \ldots, t)$. Combining (3) with (10), we deduce that

$$Y' = Y_2 \oplus Y_3 , \qquad \Omega'' = \Omega_2 \oplus \Omega_3 ,$$

$Y_3 \subset Y$, $\Omega_3 \subset \Omega$ being subspaces of dimensions $m' - t$ and $n'' - t$, respectively.

Under the above given assumptions, we state the main result of the paper.

Theorem 3.2. *Let $(D_n^{(1)})_{n\in N_0}$, $(D_n^{(2)})_{n\in N_0}$ be determinant systems for operators A_1 and A_2, respectively. Assume that*

(a) $\xi'_1, \ldots, \xi'_{n'} \in \Xi$, $y'_1, \ldots, y'_{m'-t} \in Y_3$ are such that

$$\delta' = D^{(1)}_{r'}\begin{pmatrix} \xi'_1, & & \ldots, & & \xi'_{n'} \\ y'_1, & \ldots, & y'_{m'-t}, y_1, & \ldots, & y_t \end{pmatrix} \neq 0\ ;$$

(b) $z''_1, \ldots, z''_{m''} \in Z$, $\omega''_1, \ldots, \omega''_{n''-t} \in \Omega_3$ are such that

$$\delta'' = D^{(2)}_{r''}\begin{pmatrix} \omega''_1, & \ldots, & \omega''_{n''-t}, \omega_1, & \ldots, & \omega_t \\ z''_1, & & \ldots, & & z''_{m''} \end{pmatrix} \neq 0\ ;$$

(c) $B_1 \in op(\Xi \to \Omega, Y \to X)$ is a reflexive generalized inverse of A_1 defined by the formula

$$\xi B_1 y = \frac{1}{\delta'} D^{(1)}_{r'+1}\begin{pmatrix} \xi, & \xi'_1, & & \ldots, & & \xi'_{n'} \\ y, & y'_1, & \ldots, & y'_{m'-t}, y_1, & \ldots, & y_t \end{pmatrix} \ \textit{for } (\xi, y) \in \Xi \times Y\ ; \tag{11}$$

(d) $B_2 \in op(\Omega \to \Lambda, Z \to Y)$ is a reflexive generalized inverse of A_2 defined by the formula

$$\omega B_2 z = \frac{1}{\delta''} D^{(2)}_{r''+1}\begin{pmatrix} \omega, & \omega''_1, & \ldots, & \omega''_{n''-t}, \omega_1, & \ldots, & \omega_t \\ z, & z''_1, & & \ldots, & & z''_{m''} \end{pmatrix} \ \textit{for } (\omega, z) \in \Omega \times Z\,. \tag{12}$$

Then $B_1B_2 \in op(\Xi \to \Lambda, Z \to X)$ is a reflexive generalized inverse of the operator $A_2A_1 \in op(\Lambda \to \Xi, X \to Z)$.

Proof. It follows from (11) that

$$A_1B_1 = J - \sum_{i=1}^{m'-t} y'_i \cdot \omega'_i - \sum_{i=1}^{t} y_i \cdot \omega'_{m'-t+i}\,. \tag{13}$$

Similarly, by (12)

$$B_2A_2 = J - \sum_{i=1}^{n''-t} y''_i \cdot \omega''_i - \sum_{i=1}^{t} y''_{n''-t+i} \cdot \omega_i\,. \tag{14}$$

By combining (13) with (14), we give rise to the identity

$$A_2A_1B_1B_2 = A_2B_2 - \sum_{i=1}^{m'-t} A_2 y'_i \cdot \omega'_i B_2 - \sum_{i=1}^{t} A_2 y_i \cdot \omega'_{m'-t+i} B_2\,. \tag{15}$$

In view of $A_2 y_i = 0$ $(i = 1, \dots, t)$, we transform the right-hand side of (15) into the form

$$K - \sum_{i=1}^{m''} z_i'' \cdot \lambda_i'' - \sum_{i=1}^{m'-t} A_2 y_i' \cdot \omega_i' B_2 \,. \tag{16}$$

Consequently, taking into account Lemma 3.1, we obtain

$$A_2A_1B_1B_2A_2A_1 = A_2A_1 - \sum_{i=1}^{m''} z_i'' \cdot \lambda_i'' A_2A_1 - \sum_{i=1}^{m'-t} A_2 y_i' \cdot \omega_i' B_2A_2A_1 = A_2A_1 \,. \tag{17}$$

Now, by (17), it suffices to show that B_1B_2 is $\{2\}$–inverse of A_2A_1. It follows from (15) that

$$B_1B_2A_2A_1B_1B_2 = B_1B_2 - \sum_{i=1}^{m''} B_1B_2 z_i'' \cdot \lambda_i'' - \sum_{i=1}^{m'-t} B_1B_2A_2 y_i' \cdot \omega_i' B_2 \,. \tag{18}$$

Since $A_2A_1B_1B_2$ is expressed by (16), the identities $B_2 z_i'' = 0$ $(i = 1, \dots, m'')$ yield the following form of the right-hand side of (18):

$$B_1B_2 - \sum_{i=1}^{m'-t} B_1B_2A_2 y_i' \cdot \omega_i' B_2 \,. \tag{19}$$

Next, bearing in mind (14), we present (19) in the form:

$$B_1B_2 - \sum_{i=1}^{m'-t} \sum_{j=1}^{n''-t} (\omega_i'' y_j') B_1 y_j'' \cdot \omega_i' B_2 - \sum_{i=1}^{m'-t} \sum_{j=1}^{t} (\omega_j y_i') B_1 y_{n''-t+j}'' \cdot \omega_i B_2 \,.$$

Hence, the orthogonality of ω_i'', y_j' $(i = 1, \dots, n'' - t,\ j = 1, \dots, m' - t)$ combined with (18) leads to

$$B_1B_2A_2A_1B_1B_2 = B_1B_2 - \sum_{i=1}^{m'-t} \sum_{j=1}^{t} (\omega_j y_i') B_1 y_{n''-t+j}'' \cdot \omega_i B_2 \,.$$

Since $y_{n''-t+i} \in Y_2$ $(i = 1, \dots, t)$, in view of (11), we find $B_1 y_{n''-t+i}'' = 0$. Consequently, $B_1B_2A_2A_1B_1B_2 = B_1B_2$, which completes the proof. $\square$

As a corollary to Theorem 3.2, we directly obtain the following result concerning the reverse order law for reflexive generalized inverses of Fredholm operators.

Corollary 3.3. *If $A_1 \in op(\Omega \to \Xi, X \to Y)$ and $A_2 \in op(\Lambda \to \Omega, Y \to Z)$ are Fredholm operators, then there exist $B_1 \in A_1\{1,2\}$ and $B_2 \in A_2\{1,2\}$ such that $B_1B_2 \in (A_2A_1)\{1,2\}$.*

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Received: December 09, 2016