Mathematica Aeterna, Vol. 7, 2017, no. 2, 95 - 103

On generalized b star - closed map in Topological Spaces

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Abstract

In this paper, the authors introduce a new class of generalized b star - closed map in topological spaces (briefly gbs-closed map) and study some of their properties as well as inter relationship with other closed maps.

Mathematics Subject Classification: 54C05, 54C08, 54C10.

Keywords: gbs-closed set, gb^* - closed map, b-closed map, gb-closed map, rgb-closed map and $gp*$-closed map.

1 Introduction

Different types of Closed and open mappings were studied by various researchers. In 1996, Andrijevic introduced new type of set called b-open set. A.A.Omari and M.S.M. Noorani [1] introduced and studied b-closed map. Sekar and Mariappa [10] introduced regular generalized b-closed map in topological space.

The aim of this paper is to introduce generalized b star-closed map and to continue the study of its relationship with various generalized closed maps. Through out this paper (X, τ) and (Y, σ) represent the non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned.

Let $A \subseteq X$, the closure of A and interior of A will be denoted by $cl(A)$ and $int(A)$ respectively, union of all b-open sets X contained in A is called b-interior of A and it is denoted by $bint(A)$, the intersection of all b-closed sets of X containing A is called b-closure of A and it is denoted by $bcl(A)$.

2 Preliminaries

In this section, we referred some of the closed set definitions which was already defined by various authors.

Definition 2.1. *[7] Let a subset A of a topological space (X, τ), is called a pre-open set if $A \subseteq int(cl(A))$.*

Definition 2.2. *[4] Let a subset A of a topological space (X, τ), is called a a semi-open set if $A \subseteq cl(int(A))$.*

Definition 2.3. *[7] Let a subset A of a topological space (X, τ), is called a α -open set if $A \subseteq int(cl(int(A)))$.*

Definition 2.4. *[2] Let a subset A of a topological space (X, τ), is called a b-open set if $A \subseteq cl(int(A)) \cup int(cl(A))$.*

Definition 2.5. *[3] Let a subset A of a topological space (X, τ), is called a generalized closed set (briefly g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in X.*

Definition 2.6. *[1] Let a subset A of a topological space (X, τ), is called a generalized b- closed set (briefly gb- closed) if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*

Definition 2.7. *[6] Let a subset A of a topological space (X, τ), is called a α generalized * -closed set (briefly α g*-closed) if $cl(A) \subseteq intU$ whenever $A \subseteq U$ and U is α open in X.*

Definition 2.8. *[8] Let a subset A of a topological space (X, τ), is called a g^*s-closed set (briefly $g * s$- closed) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is gs-open in X.*

Definition 2.9. *[5] Let a subset A of a topological space (X, τ), is called a regular generalized b-closed set (briefly rgb- closed) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.*

Definition 2.10. *[9] Let a subset A of a topological space (X, τ), is called a generalized b star - closed set (briefly gb^*- closed) if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is g^* open in X.*

3 On generalized b star -closed map

In this section, we introduce generalized b^* - closed map (gb^* - closed map) in topological spaces by using the notions of gb^* - closed sets and study some of their properties.

Definition 3.1. *Let X and Y be two topological spaces. A map $f : (X, \tau) \to (Y, \delta)$ is called generalized b star - closed (briefly, gb^* - closed map) if the image of every closed set in X is $gb6*$ -closed in Y.*

Theorem 3.2. *Every closed map is gb^* - closed but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ is closed map and V be an closed set in X then $f(V)$ is closed in Y. Hence gb^* - closed in Y. Then f is gb^* - closed. □

The converse of above theorem need not be true as seen from the following example.

Example 3.3. *Consider $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c, f(b) = b, f(c) = a$. The map is gb^* - closed but not closed as the image of and $\{b, c\}$ in X is $\{a, b\}$ is not closed in Y.*

Theorem 3.4. *Every semi - closed map is gb^* - closed set but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ is semi - closed map and V be an closed set in X then $f(V)$ is closed in Y. Hence gb^* - closed in Y. Then f is gb^* - closed. □

The converse of above theorem need not be true as seen from the following example.

Example 3.5. *Consider $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = b, f(b) = c, f(c) = a$. The map is gb^* - closed but not semi - closed as the image of and $\{b, c\}$ in X is $\{a, c\}$ is not semi - closed in Y.*

Theorem 3.6. *Every pre - closed map is gb^* - closed but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ be pre-closed map and V be an closed set in X then $f(V)$ is closed in Y. Hence gb^* - closed in Y. Then f is gb^* - closed. □

The converse of above theorem need not be true as seen from the following example.

Example 3.7. *Consider $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a, f(b) = b, f(c) = c$. The map is gb^* - closed but not pre - closed as the image of $\{a\}$ in X is $\{a\}$ is not pre-closed in Y.*

Theorem 3.8. *Every αg^* - closed map is gb^* - closed but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ be αg^* - closed map and V be an closed set in X then $f(V)$ is closed in Y. Hence gb^* - closed in Y. Then f is gb^* - closed. □

The converse of above theorem need not be true as seen from the following example.

Example 3.9. *Consider $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a, f(b) = b, f(c) = c$. The map is gb^* - closed but not αg^* - closed as the image of $\{b\}$ in X is $\{b\}$ is not αg^* - closed in Y.*

Theorem 3.10. *Every b - closed map is gb^* - closed but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ b - closed map and V be an closed set in X then $f(V)$ is closed in Y. Hence gb^* - closed in Y. Then f is gb^* - closed. □

The converse of above theorem need not be true as seen from the following example.

Example 3.11. *Consider $X = Y = \{a, b, c\}$,$\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c, f(b) = a, f(c) = b$. The map is gb^* -closed but not b-closed as the image of $\{b, c\}$ in X is $\{a, b\}$ is not b-closed in Y.*

Theorem 3.12. *Every g^* - closed map is gb^* - closed but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ be g^* - closed map and V be an closed set in X then $f(V)$ is closed in Y. Hence gb^* - closed in Y. Then f is gb^* - closed. □

The converse of above theorem need not be true as seen from the following example.

Example 3.13. *Consider $X = Y = \{a, b, c\}$,$\tau = \{X, \phi, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c, f(b) = a, f(c) = b$. The map is gb^* - closed but not g^* - closed as the image of and $\{b, c\}$ in X is $\{a, b\}$ is not g^* - closed in Y.*

Theorem 3.14. *Every g^* - closed map is gb^* - closed but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ be g^* closed map and V be an closed set in X then $f(V)$ is closed in Y. Hence gb^* - closed in Y. Then f is gb^* - closed. □

The converse of above theorem need not be true as seen from the following example.

Example 3.15. *Consider* $X = Y = \{a, b, c\}$,$\tau = \{X, \phi, \{a\}, \{a, c\}\}$ *and* $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = b, f(b) = a, f(c) = c$. *The map is* gb^* *- closed but not* g^*s *- closed as the image of* $\{b, c\}$ *in* X *is* $\{a, c\}$ *is not* g^*s *- closed in* Y.

Theorem 3.16. *Every* gb^* *- closed map is* rgb *- closed but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ be gb^* - closed map and V be an closed set in X then $f(V)$ is closed in Y. Hence rgb - closed in Y. Then f is rgb - closed. □

The converse of above theorem need not be true as seen from the following example.

Example 3.17. *Consider* $X = Y = \{a, b, c\}$,$\tau = \{X, \phi, \{b\}, \{a, b\}\}$ *and* $\sigma = \{Y, \phi, \{a\}\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = b, f(b) = c, f(c) = a$. *The map is* rgb *- closed but not* gb^* *- closed as the image of* $\{a, c\}$ *in* X *is* $\{a, b\}$ *is not* gb^**-closed in* Y.

Theorem 3.18. *A map* $f : (X, \tau) \to (Y, \sigma)$ *is continuous and* gb^* *- closed set* A *is* gb^* *-closed set of* X *then* $f(A)$ *is* gb^* *closed in* Y.

Proof. Let $f(A) \subseteq U$ where U is g^* open set in Y. Since f is continuous, $f^{-1}(U)$ is open set containing A. Hence $bcl(A) \subseteq f^{-1}(U)$ (as A is gb^* - closed). Since f is gb^* - closed $f(bcl(A)) \subseteq U$ is gb^* closed set $\Rightarrow bcl(f(bcl(A)) \subseteq U$, Hence $bcl(A) \subseteq U$. So that $f(A)$ is gb^* - closed set in Y. □

Theorem 3.19. *If a map* $f : (X, \tau) \to (Y, \sigma)$ *is continuous and closed set and* A *is* gb^* *- closed then* $f(A)$ *is* gb^* *- closed in* Y.

Proof. Let F be a closed set of A then F is gb^* - closed set. By theorem 3.18 $f(A)$ is gb^* - closed. Hence $f_A(F) = f(F)$ is gb^* - closed set of Y. Here f_A is gb^* - closed and also continuous. □

Theorem 3.20. *If* $f : (X, \tau) \to (Y, \sigma)$ *is closed map and* $g : (Y, \sigma) \to (Z, \eta)$ *is* gb^* *- closed map , then the composition* $g \cdot f : (X, \tau) \to (Z, \eta)$ *is* gb^* *- closed map.*

Proof. Let F be any closed set in (X, τ). Since f is closed map, $f(F)$ is closed set in (Y, σ). Since g is gb^* - closed map, $g(f(F))$ is gb^* - closed set in (Z, η). That is $g \cdot f(F) = g(f(F))$ is gb^* closed. Hence $g \cdot f$ is gb^* closed map. □

Remark 3.21. *If* $f : (X, \tau) \to (Y, \sigma)$ *is* gb^* *- closed map and* $g : (Y, \sigma) \to (Z, \eta)$ *is closed map, then the composition need not* gb^* *- closed map .*

Theorem 3.22. *A map* $f : (X, \tau) \to (Y, \sigma)$ *is* gb^* *- closed if and only if for each subset* S *of* (Y, σ) *and each open set* U *containing* $f^{-1}(S) \subset U$, *there is a* gb^* *- open set* V *of* (Y, σ) *such that* $S \subset V$ *and* $f^{-1}(V) \subset U$.

Proof. Suppose f is gb^* - closed. Let $S \subset Y$ and U be an open set of (X, τ) such that $f^{-1}(S) \subset U$. Now $X - U$ is closed set in (X, τ). Since f is gb^* - closed, $f(X - U)$ is gb^* - closed set in (Y, σ). There fore $V = Y f(X - U)$ is an gb^* - open set in (Y, σ). Now $f^{-1}(S) \subset U$ implies $S \subset V$ and $f^{-1}(V) = X - f^{-1}(f(X - U)) \subset X - (X - V) = U$. (ie) $f^{-1}(V) \subset U$.

Conversely,

Let F be a closed set of (X, τ). Then $f^{-1}(f(F^c)) \subset F^c$ and F^c is an open in (X, τ). By hypothesis, there exist a gb^* - open set V in (Y, σ) such that $f(F^c) \subset V$ and $f^{-1}(V) \subset F^c \Rightarrow F \subset f^{-1}(V)^c$. Hence $V^c \subset f(F) \subset f(((f^{-1}(V))^c)^c \subset V^c \Rightarrow f(V) \subset V^c$. Since V^c is gb^* - closed, $f(F)$ is gb^* - closed. (ie) $f(F)$ is gb^* - closed in Y. Therefore f is gb^* - closed. □

Theorem 3.23. *If $f : X_1 \times X_2 \to Y_1 \times Y_2$ is defined as $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$, then $f : X_1 \times X_2 \to Y_1 \times Y_2$ is gb^* closed map.*

Proof. Let $U_1 \times U_2 \subset X_1 \times X_2$ where $U_i \in pgbcl(X_i)$, for $i = 1, 2$. Then $f(U_1 \times U_2) = f_1(U_1) \times f_2(U_2) \in pgbcl(X_1 \times Y_2)$. Hence f is gb^* - closed set. □

Theorem 3.24. *Let $h : X \to X_1 \times X_2$ be gb^* - closed map and Let $f_i : X \times X_i$ be define as $h(x) = (x_1, x_2)$ and $f_i(x) = x_i$, then $f_i : X \times X_i$ is gb^* - closed map for $i = 1, 2$.*

Proof. Let $U_1 \times U_2 \in X_1 \times X_2$, then $f_1(U_1) = h_1(U_1 \times X_2) \in gb^*cl(X)$, there fore f_1 is gb^* - closed. Similarly we have f_2 is gb^* - closed. Thus f_i is gb^* - closed map for $i = 1, 2$. □

Theorem 3.25. *For any bijection map $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*
(i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is gb^* - continuous.
(ii) f is gb^* - open map.
(iii) f is gb^* - closed map.

Proof. (i)⇒(ii) Let U be an open set of (X, τ). By assumption, $(f^{-1})^{-1}(U) = f(U)$ is gb^* - open in (Y, σ) and so f is gb^* - open.

(ii)⇒(iii) Let F be a closed set of (X, τ). Then F^c is open set in (X, τ). By assumption $f(F^c)$ is gb^* - open in (Y, σ). Therefore $f(F^c) = f(F)^c$ is gb^* - open in (Y, σ). That is $f(F)$ is gb^* - closed in (Y, σ). Hence f is gb^* - closed.

(iii)⇒(i) Let F be a closed set of (X, τ). By assumption, $f(F)$ is gb^* - closed in (Y, σ). But $f(F) = (f^{-1})^{-1}(F) \Rightarrow (f^{-1})$ is continuous. □

Remark 3.26. *The following examples show that gb^* - closed and gb closed maps are independent.*

Example 3.27. *Let $X = Y = \{a, b, c\}$ with $\tau = \{X, \varphi, \{b\}, \{b, c\}\}$ and $\sigma = \{Y, \varphi, \{a\}, \{a, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$, then f is gb closed map but not gb^* closed map as the image of and $\{a, c\}$ in X is $\{a, b\}$ is not gb^* closed set in Y.*

Example 3.28. *Let* $X = Y = \{a, b, c\}$ *with* $\tau = \{X, \varphi, \{a\}, \{a, b\}\}$ *and* $\sigma = \{Y, \varphi, \{c\}, \{b, c\}\}$. *Define a function* $f : (X, \tau) \to (Y, \sigma)$ *by* $f(a) = c$, $f(b) = a$, $f(c) = b$, *then* f *is* gb^* *closed but not gb closed as the image of and* $\{b, c\}$ *in* X *is* $\{a, b\}$ *is not gb closed set in* Y.

4 On generalized b star -open map

In this section, we introduce generalized b star- open map (briefly gb^* - open) in topological spaces by using the notions of gb^* - open sets and study some of their properties.

Definition 4.1. *Let* X *and* Y *be two topological spaces. A map* $f : (X, \tau) \to (Y, \delta)$ *is called generalized b star - open (briefly,* gb^* *- open) if the image of every open set in* X *is* gb^* *- open in* Y.

Theorem 4.2. *Every open map is* gb^* *- open but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \delta)$ is open map and V be an open set in X then $f(V)$ is open in Y. Hence gb^* - open in Y. Then f is gb^* - open. □

The converse of above theorem need not be true as seen from the following example.

Example 4.3. *Consider* $X = Y = \{a, b, c\}$,$\tau = \{X, \phi, \{c\}, \{a, c\}\}$ *and* $\sigma = \{Y, \phi, \{b\}\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = b, f(b) = c, f(c) = a$. *The map is* gb^* *- open but not open as the image of and* $\{a, c\}$ *in* X *is* $\{b, c\}$ *is not open in* Y.

Theorem 4.4. *A map* $f : (X, \tau) \to (Y, \sigma)$ *is* gb^* *- closed set if and only if for each subset* S *of* Y *and for each open set* U *containing* $f^{-1}(S) \subset U$ *there is a* gb^* *- open set* V *of* Y *such that* $S \subset U$ *and* $f^{-1}(V) \subset U$.

Proof. Suppose f is gb^* - closed set. Let $S \subset Y$ and U be an open set of (X, τ) such that $f^{-1}(S) \subset U$. Now $X - U$ is closed set in (X, τ). Since f is gb^* closed, $f(X - U)$ is gb^* closed set in (Y, σ). Then $V = Y - f(X - U)$ is gb^* open set in (Y, σ). There fore $f^{-1}(S) \subset U$ implies $S \subset V$ and $f^{-1}(V) = X - f^{-1}(f(X - U)) \subset X - (X - V) = U$. (ie) $f^{-1}(V) \subset U$.

Conversely,

Let F be a closed set of (X, τ). Then $f^{-1}(f(F^c)) \subset F^c$ and F^c is an open in (X, τ). By hypothesis, there exists a gb^* open set V in (Y, σ) such that $f(F^c) \subset V$ and $f^{-1}(V) \subset F^c \Rightarrow F \subset (f^{-1}(V))^c$. Hence $V^c \subset f(F) \subset f(((f^{-1}(V))^c)^c) \subset V^c \Rightarrow f(V) \subset V^c$. Since V^c gb^* - closed, $f(F)$ is gb^*-closed. (ie) $f(F)$ is gb^* - closed in (Y, σ) and there fore f is gb^* - closed. □

Theorem 4.5. *For any bijection map $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent.*
(i) $f^{-1} : (X, \tau) \to (Y, \sigma)$ is gb^* - continuous.
(ii) f is gb^* open map.
(iii) f is gb^* - closed map.

Proof. (i)$\Rightarrow$ (ii) Let U be an open set of (X, τ). By assumption $(f^{-1})^{-1}(U) = f(U)$ is gb^* - open in (Y, σ). There fore f is gb^* - open map.

(ii)$\Rightarrow$ (iii) Let F be closed set of (X, τ), Then F^c is open set in (X, τ). By assumption, $f(F^c)$ is gb^* - open in (Y, σ). There fore $f(F)$ is gb^* - closed in (Y, σ). Hence f is gb^*- closed.

(iii)$\Rightarrow$ (i) Let F be a closed set of (X, τ), By assumption $f(F)$ is gb^* - closed in (Y, σ). But $f(F) = (f^{-1})^{-1}(F)$. Hence $f^{-1} : (X, \tau) \to (Y, \sigma)$ is gb^* -continuous. □

5 Conclusion

The classes of generalized b star -closed map and generalized b star -open map defined using gb^* -closed sets form a topology. The gb^*-closed maps can be used to derive a new decomposition of continuity, contra continuous function, almost contra continuous function, closure and interior. This idea can be extended to fuzzy topological space and fuzzy intuistic topological spaces.

Acknowledgment

The authors gratefully acknowledge the Dr. G. Balaji, Professor of Mathematics & Head, Department of Science & Humanities, Al-Ameen Engineering College, Erode - 638 104, for encouragement and support. The authors also heartfelt thank to Dr. M. Vijayarakavan, Associate Professor, Department of Mathematics, VMKV Engineering College, Salem 636 308, Tamil Nadu, India, for his kind help and suggestions.

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Received: March 18, 2017