

Generalizations of the Stone — Weierstrass theorem

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Abstract

The classical Stone—Weierstrass theorem is generalized in two directions. In the first of them, the assumption of compactness of the domain of definition is weakened to countable compactness, and in the second, we waive topological assumptions at all, due to substitution of a topology by a convergence, and thereon the topological continuity by the sequential one.

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Introduction

In this article, the classical Stone – Weierstrass theorem is generalized in two directions. In Section 1, the compactness assumption is weakened, and in Section 2 topological assumptions are waived at all, due to substitution of a topology in the domain of definition of functions by a convergence, and thereon the topological continuity by the sequential one. Thus modified formulation appears natural in the situation when we proceed from some a priori convergence (say the ordinal convergence in an ordered set, in particular the dominated pointwise convergence of functions) and are not interested in the topology – either generating this convergence or generated by it. This situation is illustrated in Example 2.12.

1 Generalization for functions on a countably compact topological space

Recall that a vector space over a field \mathbb{K} is called *linear ring* (or *algebra*, but this term is ambiguous) over this field if it is equipped with an associative multiplication that is distributive w. r. t. the addition and obeys the law $(\alpha x)y = \alpha(xy) = x(\alpha y)$, where α is a scalar, and x, y are vectors. A linear ring containing unity is called *unital*.

By $C(T)$ we denote the set of all complex-valued continuous functions on a topological space T .

The Stone – Weierstrass theorem has two versions: for real and for complex functions. The latter reads (see, e. g., Theorem IV.6.17 in [2]):

Let T be a compact topological space, and \mathfrak{B} be a unital linear subring of $C(T)$. Suppose that \mathfrak{B} separates points of T and contains the conjugate of every its element. Then \mathfrak{B} is dense in $C(T)$.

In this theorem, the Hausdorff property of T need not be postulated (as is usually done), because it ensues, as one can easily see, from the separation condition.

Recall that a set in a topological set is called *countably compact* if every its countable open covering contains a finite subcovering. The following two statements are well-known.

Lemma 1.1. *Let f be a continuous mapping of a countably compact topological space T into some topological space. Then the set $\text{im } f$ is also countably compact.*

Lemma 1.2. *In order that a topological space T be countably compact it is necessary and sufficient that every decreasing sequence of its non-void closed subsets have a non-void intersection.*

Corollary 1.3 (of Lemma 1.1). *Every continuous function on a countably compact topological space is bounded and attains both its exact bounds.*

Corollary 1.3 allows us to endow $C(T)$ with the uniform norm $\|x\| = \max_{t \in T} |x(t)|$, thus converting it into a Banach space (its completeness is a familiar fact — see, e. g., Section IV.6 in [2]). So $C(T)$ is a Banach algebra.

Proposition 1.4. *Let T be a countably compact topological space such that $C(T)$ separates its points. Then every maximal ideal in $C(T)$ is of the form $M_s = \{x(\cdot) : x(s) = 0\}$.*

Proof. If a continuous function on T does not vanish at any point, then by Corollary 1.3 both its exact bounds are nonzero. Hence this function is an invertible element of the linear ring $C(T)$ and, consequently, cannot belong to

any proper ideal. So any maximal (and therefore, by definition, proper) ideal consists only of functions vanishing at least at one point. Obviously, all the ideals M_s are maximal.

Let t_1 and t_2 be two different points of T . The assumption that $C(T)$ separates points of T implies that

$$M_{t_i} \setminus M_{t_{3-i}} \neq \emptyset, \quad i = 1, 2. \tag{1}$$

If I is an ideal such that $x(t_1) = 0 = x(t_2)$ for all $x \in I$, then $I \subset M_{t_1} \cap M_{t_2}$, which together with (1) implies that I is a proper subset of both M_{t_1} and M_{t_2} . So I is not maximal. \square

Note that the above proof has few common with that for compact T (see, e. g., Example 11.13a in [5]).

Let $\mathcal{M}_{\mathfrak{A}}$ denote the set of all characters (linear multiplicative functionals) on a Banach algebra \mathfrak{A} (in case $\mathfrak{A} = C(T)$ we write simply \mathcal{M}). The expression $\chi(x)$, where $\chi \in \mathcal{M}_{\mathfrak{A}}$ and $x \in \mathfrak{A}$, will be otherwise written as $\widehat{x}(\chi)$, so that \widehat{x} is a linear functional on $\mathcal{M}_{\mathfrak{A}}$. As usually, we equip $\mathcal{M}_{\mathfrak{A}}$ with Tikhonov's topology. As is easily seen, it coincides with the topology induced by all \widehat{x} . It is well known that the topological space $\mathcal{M}_{\mathfrak{A}}$ is Hausdorff and compact (see [3, Lemma IX.2.8] or [5, Theorem 11.9a]), so one can endow $C(\mathcal{M}_{\mathfrak{A}})$ with the uniform norm. The first Gelfand – Neumark theorem ([3, Lemma IX.3.6], [5, Theorem 11.18]) states that, for any commutative C^* -algebra \mathfrak{A} (in particular, for $\mathfrak{A} = C(T)$) the mapping $x \mapsto \widehat{x}$ is an isometrical isomorphism of \mathfrak{A} onto $C(\mathcal{M}_{\mathfrak{A}})$.

Theorem 1.5. *Let T be a countably compact topological space, and \mathfrak{B} be a unital linear subring of the linear ring $C(T)$. Suppose that \mathfrak{B} separates points of T and contains the conjugate of every its element. Then \mathfrak{B} is dense in $C(T)$.*

Proof. Denote $\widehat{\mathfrak{B}} = \{\widehat{x}, x \in \mathfrak{B}\} (\subset C(\mathcal{M}))$. Obviously, this is a linear ring. The C^* -algebras $C(T)$ and $C(\mathcal{M})$ being isometric, it suffices to show that $\widehat{\mathfrak{B}}$ is dense in $C(\mathcal{M})$. Gelfand's theorem on the one-to-one correspondence between characters and maximal ideals ([3, Lemma IX.2.1], [5, Theorem 11.5]) and Proposition 1.4 imply together that $\mathcal{M} = \{\delta_s, s \in T\}$, where $\delta_s x = x(s)$. Let $\delta_{t_1} \neq \delta_{t_2}$. Then, obviously, $t_1 \neq t_2$, whence by the separation assumption there exists $x \in \mathfrak{B}$ such that $x(t_1) \neq x(t_2)$, or, the same, $\widehat{x}(\delta_{t_1}) \neq \widehat{x}(\delta_{t_2})$. Thus $\widehat{\mathfrak{B}}$ separated points of \mathcal{M} .

The ring \mathfrak{B} being unital, so is $\widehat{\mathfrak{B}}$. Its unity is $\widehat{1}$ – the functional equal to 1 on all characters.

Furthermore, for any $x \in C(T)$ $\overline{\widehat{x}} = \widehat{\bar{x}}$ (the bar signifies complex conjugation) by Theorem 11.18 in [5]. Thus $\widehat{\mathfrak{B}}$ contains the conjugate of every its element.

Now, denseness of $\widehat{\mathfrak{B}}$ in $C(\mathcal{M})$ follows from compactness of \mathcal{M} and the Stone – Weierstrass theorem. \square

2 Generalizations for functions on a sequentially compact convergence space

In what follows, T is an arbitrary (until additional assumptions are imposed) non-void set, and ι is a mapping of T into $2^{T^{\mathbb{N}}} \setminus \{\emptyset\}$ (i. e., for each $t \in T$ $\iota[t]$ is a non-void set of sequences in T).

We will say that a set $A \subset T$ is ι -closed if for any $t \in T$ the relation

$$A^{\mathbb{N}} \cap \iota[t] \neq \emptyset \quad (2)$$

implies that $t \in A$. According to this definition the whole set T is always ι -closed.

Lemma 2.1. *The intersection of an arbitrary collection of ι -closed sets is a ι -closed set.*

Proof. Let $A = \bigcap A_{\alpha}$, where α runs through some set of tags and each A_{α} is a ι -closed subset of T . Then $A^{\mathbb{N}} = \bigcap A_{\alpha}^{\mathbb{N}}$, so one can rewrite (2) in the form

$$A_{\alpha}^{\mathbb{N}} \cap \iota[t] \neq \emptyset \text{ for all } \alpha.$$

Hence by the assumption on A_{α} we have $t \in A_{\alpha}$ for all α . \square

Lemma 2.1 enables us to define the ι -closure of a set $V \subset T$ as the smallest of ι -closed sets containing V . We say that a function x on T is ι -continuous if $x(t_n) \rightarrow x(t)$ for every $t \in T$ and $(t_n) \in \iota[t]$.

By τ_{ι} we denote the collection of subsets of T that consists of T and the complements to all ι -closed sets. We will say, following Fréchet [4], that ι is a *convergence* (keeping in mind “in T ”) if for every $t \in T$ it contains all subsequences of any sequence $(t_n) \in \iota[t]$ (Fréchet postulated one more property which in the subsequent theorems will ensue from the separation condition — see Remark 2.11 below). In this case, the pair (T, ι) will be called a *convergence space*, and the prefix ι - will be replaced with the adjective ‘sequential’. We say that a sequence (t_n) in such a space *converges* to a point $t \in T$ if $(t_n) \in \iota[t]$, which will be otherwise written as $t_n \rightarrow t$ (this, of course, does not exclude that $t_n \rightarrow t'$ for some $t' \neq t$.) Thus a function x on a convergence space is, by the definition given in the previous paragraph, *sequentially continuous* if the relation $t_n \rightarrow t$ in the domain of definition entails $x(t_n) \rightarrow x(t)$. The family of all \mathbb{C} -valued sequentially continuous functions on T will be denoted by $C_{\text{seq}}(T)$.

Lemma 2.2. *Let ι be a convergence in T . Then the union of any two ι -closed sets is ι -closed, too.*

Proof. Let $(t_n) \in (F_1 \cup F_2)^\mathbb{N}$. If F_k contains infinitely many t_n 's (by the choice of the sequence this is true either for $k = 1$ or for $k = 2$) and is ι -closed, then F_k (and all the more $F_1 \cup F_2$) contains every point t such that $t_n \rightarrow t$. \square

Combining Lemmas 2.1 and 2.2, we get

Corollary 2.3. *Let ι be a convergence in T . Then τ_ι is a topology on T .*

Lemma 2.4. *Let ι be a convergence in T . Then every sequentially continuous function on T is continuous w. r. t. the topology τ_ι .*

Proof. Let $x \in C_{seq}(T)$ and F be a closed set in \mathbb{C} . We have to show that its pre-image $x^{-1}(F)$ is ι -closed.

By the choice of x , for an arbitrary sequence $(t_n) \in T^\mathbb{N}$ quasi-converging to some t , one has $x(t_n) \rightarrow x(t)$. If herein $x(t_n) \in F$, $n \in \mathbb{N}$, then, due to closedness of F , $x(t) \in F$. \square

We will say that a convergence space is *sequentially compact* if every sequence of its points contains a convergent subsequence. Since the topological convergence is, obviously, a particular case of the axiomatic one, this definition concerns topological spaces, as well. For them, it takes the form adopted in topology.

The following (familiar) statement is immediate from Lemma 1.2.

Lemma 2.5. *Every sequentially compact topological space is countably compact.*

We say that a convergence space (T, ι) is *sequentially Hausdorff* if for any $(t_n) \in T^\mathbb{N}$ there exists at most one $t \in T$ such that $(t_n) \in \iota[t]$. Sequentially compact sequentially Hausdorff convergence spaces will be called briefly SCH-spaces.

Lemma 2.6. *Let a sequence (t_n) in an SCH-space (T, ι) converge to some t . Then it converges to t in the topological space (T, τ_ι) .*

Proof. Take an arbitrary τ_ι -neighborhood U of t and denote $F = T \setminus U$. This is a sequentially closed set by the construction of τ_ι . We have to show that F does not contain eventually all t_n 's.

Suppose it is wrong, i. e., there exists an infinite set $J_0 \subset \mathbb{N}$ such that $t_n \in F$ for all $n \in J_0$. Sequential compactness of (T, ι) implies existence of an infinite set $J \subset J_0$ and a point $t' \in T$ such that $t_n \rightarrow t'$ as $n \rightarrow \infty$, $n \in J$ (and therefore $t' \in F$). On the other hand, the sequential Hausdorff property of the space implies that $t' = t$ and therefore $t' \in U$. But $U \cap F = \emptyset$. \square

Corollary 2.7. *Let (T, ι) be an SCH-space. Then the topological space (T, τ_ι) is sequentially compact.*

Lemma 2.5 and Corollary 2.7 yield together

Corollary 2.8. *Let (T, ι) be an SCH-space. Then the topological space (T, τ_ι) is countably compact.*

To avoid misunderstanding of the final results we stress that the symbol T in the notation $C_{seq}(T)$ means a convergence space (T, ι) , whereas in $C(T)$ it stands for the associated topological space (T, τ_ι) .

Theorem 2.9. *Let (T, ι) be an SCH-space, and \mathfrak{B} be a unital linear subring of $C_{seq}(T)$. Suppose that \mathfrak{B} separates points of T and contains constant functions and the conjugate of every its element. Then \mathfrak{B} is dense in $C(T)$.*

Proof. Lemma 2.4 asserts that $C_{seq}(T) \subset C(T)$, so \mathfrak{B} satisfies all the conditions of Theorem 1.5. So does T due to Corollary 2.8. Consequently, the conclusion of Theorem 1.5 is valid. \square

Remark 2.10. Lemma 2.4 allows to substitute $C(T)$ in Theorems 2.9 by $C_{seq}(T)$, thus somewhat weakening the statement, but excluding the topology from its assertion.

Remark 2.11. The assumption that \mathfrak{B} (and all the more $C_{seq}(T)$) contains unity implies, obviously, that for any $t \in T$ the stationary sequence (t, t, \dots) converges to t . So we need not postulate this property of convergence, once we use this notion only in the theorems where the separation property is a condition.

Example 2.12. Let $\mathbb{M}(X)$ and $C_b(X)$ denote the set of all signed measures on the Baire σ -algebra in a topological space X and the set of all bounded continuous complex-valued functions on X , respectively. Recall that the *weak convergence* of a sequence (μ_n) in $\mathbb{M}(X)$ to $\mu \in \mathbb{M}(X)$ means, by Definition 8.1.1 in [1] (introduced by Alexandroff), that $\int f d\mu_n \rightarrow \int f d\mu$ for every $f \in C_b(X)$. This definition is formulated irrespectively of any topology, so it is natural to consider $\mathbb{M}(X)$ equipped with the weak convergence as a convergence space. This space is sequentially Hausdorff, since every $\mu \in \mathbb{M}(X)$ is uniquely determined by the values of the integrals $\int f d\mu$, $f \in C_b(X)$ [1, Theorem 7.10.1].

We regard vectors from \mathbb{R}^d as columns. Then elements of the dual space \mathbb{R}^{d*} will be rows. Recall that the *characteristic function* $\mu^\#$ of a signed measure $\mu \in \mathbb{M}(\mathbb{R}^d)$ is defined by $\mu^\#(z) = \int e^{izx} \mu(dx)$, where z ranges over \mathbb{R}^{d*} . For fixed z , it is a function of μ . We denote this function by φ_z . In other words,

$\varphi_z(\mu) = \mu^\#(z)$. It is well known (see, e. g., Proposition 3.8.6 in [1]) that every $\mu \in \mathbb{M}(\mathbb{R}^d)$ is uniquely determined by its characteristic function. This means that the functions φ_z , $z \in \mathbb{R}^{d*}$, separate points of $\mathbb{M}(\mathbb{R}^d)$. Each of them is sequentially continuous by the very definition of the weak convergence. Furthermore, $\varphi_0(\mu) = \mu(\mathbb{R}^d) = \text{const}$ and $\varphi_{-z} = \overline{\varphi_z}$. Thus Theorem 2.9 asserts the following: for any sequentially compact set $T \subset \mathbb{M}(\mathbb{R}^d)$, function $\Lambda \in C(T)$ and positive number ε , there exist natural numbers n, m_1, \dots, m_n , complex numbers c_1, \dots, c_n and vectors $z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n} \in \mathbb{R}^{d*}$ such that

$$\left| \Lambda(\mu) - \sum_{k=1}^n c_k \prod_{j=1}^{m_k} \mu^\#(z_{kj}) \right| < \varepsilon$$

for all $\mu \in T$.

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