Mathematica Aeterna , Vol. 7, 2017, no. 2, 177 - 184

On the oscillation of even order neutral delay differential equations

Shaoqin Gao
College of Mathematics and Information Science,
Hebei University, Baoding, 071002, China

Zimeng Chen
School of Management Hebei University, Baoding, 071002, China

Wenying Shi
College of Mathematics and Information Science,
Hebei University, Baoding, 071002, China

Abstract

In this paper, we study the oscillatory behavior of the following even order neutral delay differential equation

$$(r(t)((x(t)+p(t)x(\tau(t)))^{(n-1)})^{\alpha})' + q(t)x^{\alpha}(\tau(t)) = 0, t \geq t_0.$$

We give some sufficient conditions for oscillation of this equation using Riccati transformation technique . The results obtained extend some of the known results. An example is given to illustrate the main results.
Keywords: Oscillatory solution; Neutral delay differential equation; Even order.
Mathematics Subject Classification: 34C10, 34K11.

1 Introduction

In this paper, we are concerned with the oscillatory behavior of the solution of the even order neutral delay differential equation

$$(r(t)((x(t)+p(t)x(\tau(t)))^{(n-1)})^{\alpha})' + q(t)x^{\alpha}(\tau(t)) = 0, t \geq t_0. \tag{1.1}$$

where n is even, α is the ratio of odd positive integers, $q(t) \in C([t_0,\infty)), r(t), p(t), \tau(t) \in C^1([t_0,\infty))$ and

(H_1) $q(t) \geq 0, 0 \leq p(t) < 1;$

(H_2) $r(t) > 0, r'(t) \geq 0, \int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)} dt = \infty;$

(H_3) $\tau(t) \leq t, \lim_{t\to\infty} \tau(t) = \infty.$

We set $z(t) = x(t)+p(t)x(\tau(t))$. By a solution of equation (1.1), we mean a function $x(t) \in C([t_0,\infty))$, such that $z(t) \in C^{n-1}([t_0,\infty))$ and $r(t)(z^{(n-1)}(t))^{\alpha} \in C^1([t_0,\infty))$ and $x(t)$ satisfies (1.1) on $[t_0,\infty)$. We consider only those solutions $x(t)$ of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_0$. We assume that (1.1) possesses such solution. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$; otherwise, it is called non-oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

The oscillatory behavior of n order differential equation has been the subject of intensive research [2-10] and therein. In [5] Grace and Lalli gave the oscillation criteria of even-order equation

$$x^{(n)}(t) + q(t)x(\tau(t)) = 0.$$

In $[4, 6-9]$ the authors studied the oscillatory behavior of high-order differential equation

$$(r(t)(x^{(n-1)}(t))^{\alpha})' + q(t)x^{\alpha}(\tau(t)) = 0.$$

The authors in [2] studied the oscillation of an even-order neutral differential equation

$$[x(t) + p(t)x(\tau(t))]^{(n)} + q(t)x(\sigma(t)) = 0, t \geq t_0.$$

We develop theorems related the oscillatory behavior and provide sufficient conditions for the equation (1.1) to be oscillatory. We give some sufficient conditions for oscillation of equation (1.1) using Riccati transformation technique. Our results obtained extend some of the known results mentioned above.

In the following, all occurring functional inequalities considered in this paper are assumed to be hold eventually, that is, they are satisfied for all t large enough.

2 Main Results

In this section, we will establish some oscillation criteria for (1.1) using Riccati transformation and we will give the oscillation property based on the comparison theorem. We begin with the following three Lemmas.

Lemma 2.1 (*See*[10]). *Let $f(t) \in C^n([t_0,\infty),(0,\infty))$. If $f^{(n)}(t)$ is eventually of one sign for all large t, then there exist a $t_x \geq t_0$ and an integer l, $0 \leq l \leq n$ with $n+l$ even for $f^{(n)}(t) \geq 0$, or $n+l$ odd for $f^{(n)}(t) \leq 0$ such that $l > 0$ yields $f^{(k)}(t) > 0$ for $t \geq t_x, k = 0, 1, \cdots, l-1$, and $l \leq n-1$ yields $(-1)^{l+k} f^{(k)}(t) > 0$ for $t \geq t_x, k = l, l+1, \cdots, n-1$.*

Lemma 2.2 (*See*[3, *Lemma*2.2.3]). *Let* $f(t) \in C^n([t_0,\infty),\mathrm{R}^+)$. *Assume that* $f^{(n)}(t)$ *is of fixed sign and not identically zero on a subray of* $[t_0,\infty)$, *and there exists a* $t_1 \geq t_0$ *such that* $f^{(n-1)}(t)f^{(n)}(t) \leq 0$ *for all* $t \geq t_1$. *If* $\lim_{t\to\infty} f(t) \neq 0$, *then for every* $\lambda \in (0,1)$, *there exists* $t_\lambda \in [t_1,\infty)$ *such that*

$$f(t) \geq \frac{\lambda}{(n-1)!}t^{n-1}|f^{(n-1)}(t)|$$

holds on $[t_\lambda,\infty)$.

Lemma 2.3 (*See*[11]). *If a function* $y(t)$ *satisfies* $y^{(i)}(t) > 0, i = 0,1,\cdots,k$ *and* $y^{(k+1)}(t) \leq 0$, *then* $y(t)/y'(t) \geq t/k$ *eventually.*

Now, we present the main results. For the sake of convenience, we use the notation $\rho'_+(t) := max(0,\rho'(t))$, $\delta'_+(t) := max(0,\delta'(t))$, and $Q(\eta) := \int_\eta^\infty q(s)(1-p(\tau(s)))^\alpha(\frac{\tau(s)}{s})^\alpha ds.$

Theorem 2.4 *Let* $n \geq 4$ *be even and* $(H_1)-(H_3)$ *hold. Assume that there exist two functions* $\rho(t),\delta(t) \in C^1([t_0,\infty),(0,\infty))$ *such that for some constant* $\lambda_0 \in (0,1)$,

$$\int^\infty \left[\rho(t)q(t)(1-p(\tau(t)))^\alpha(\frac{\tau^\alpha(t)}{t^\alpha})^{n-1} - r(t)(\frac{\rho'_+(t)}{\alpha+1})^{\alpha+1}(\frac{(n-2)!}{\lambda_0\rho(t)t^{n-2}})^\alpha\right]dt = \infty, \tag{2.1}$$

and either

$$\int^\infty q(s)(1-p(\tau(s)))^\alpha(\frac{\tau(s)}{s})^\alpha ds = \infty, \tag{2.2}$$

or

$$\int^\infty \eta^{n-4}Q^{\frac{1}{\alpha}}(\eta)r^{\frac{-1}{\alpha}}(\eta)d\eta = \infty, \tag{2.3}$$

or

$$\int^\infty \left[\frac{\delta(t)}{(n-4)!}\int_t^\infty (\eta-t)^{n-4}Q^{\frac{1}{\alpha}}(\eta)r^{\frac{-1}{\alpha}}(\eta)d\eta - \frac{\delta'_+(t)^2}{4\delta(t)}\right]dt = \infty. \tag{2.4}$$

Then (1.1) *is oscillatory.*

Proof. Assume that (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is eventually positive. It follows from (1.1), (H_1), (H_2) and Lemma 2.1 that there exist two possible cases for $t \geq t_1 \geq t_0$ large enough:

(1) $z(t) > 0, z'(t) > 0, z''(t) > 0, z^{(n-1)}(t) > 0, (r(t)(z^{(n-1)}(t))^\alpha)' \leq 0;$

(2)$z(t) > 0, z^{(j)}(t) > 0, z^{(j+1)}(t) < 0$ for every odd number $j \in \{1,2,\cdots,n-3\}$, $z^{(n-1)}(t) > 0$, and $(r(t)(z^{(n-1)}(t))^\alpha)' \leq 0$.
We consider each of two cases separately.

Assume that case (1) holds. We know that $\lim_{t\to\infty} z'(t) \neq 0$. By virtue of Lemma 2.2, for every constant $\lambda \in (0,1)$ and for all large t, we get

$$z'(t) \geq \frac{\lambda}{(n-2)!} t^{n-2} z^{(n-1)}(t). \tag{2.5}$$

Noting $z'(t) > 0$ and (H_3), we have

$$x(t) \geq (1-p(t))z(t).$$

By (1.1) we get

$$(r(t)(z^{(n-1)}(t))^{\alpha})' \leq -q(t)(1-p(\tau(t)))^{\alpha} z^{\alpha}(\tau(t)). \tag{2.6}$$

By Lemma 2.3, we obtain

$$\frac{z(t)}{z'(t)} \geq \frac{t}{n-1}.$$

So $z(t)/t^{n-1}$ is non-increasing and

$$\frac{z(\tau(t))}{z(t)} \geq \frac{\tau^{n-1}(t)}{t^{n-1}}. \tag{2.7}$$

Now we introduce a Riccati substitution

$$u(t) := \rho(t)\frac{r(t)z^{(n-1)}(t)^{\alpha}}{z^{\alpha}(t)}, t \geq t_1. \tag{2.8}$$

Then $u(t) > 0$ on $[t_1, \infty)$. Following from (2.5), (2.6) and (2.7) we have

$$u'(t) = \rho'(t)\frac{r(t)z^{(n-1)}(t)^{\alpha}}{z^{\alpha}(t)} + \rho(t)\frac{(r(t)z^{(n-1)}(t)^{\alpha})'}{z^{\alpha}(t)} - \alpha\rho(t)\frac{r(t)z^{(n-1)}(t)^{\alpha}z'(t)}{z^{\alpha+1}(t)}$$

$$\leq -\rho(t)q(t)(1-p(\tau(t)))^{\alpha}(\frac{\tau^{n-1}(t)}{t^{n-1}})^{\alpha} + \rho'(t)\frac{r(t)z^{(n-1)}(t)^{\alpha}}{z^{\alpha}(t)} - \frac{\alpha\lambda t^{n-2}}{(n-2)!}\frac{\rho(t)r(t)z^{(n-1)}(t)^{\alpha+1}}{z^{\alpha+1}(t)}.$$

By virtue of (2.8), we have

$$u'(t) \leq -\rho(t)q(t)(1-p(\tau(t)))^{\alpha}(\frac{\tau^{n-1}(t)}{t^{n-1}})^{\alpha} + \frac{\rho'_+(t)}{\rho(t)}u(t) - \frac{\alpha\lambda t^{n-2}}{(n-2)!\rho^{\frac{1}{\alpha}}(t)r^{\frac{1}{\alpha}}(t)}u^{\frac{\alpha+1}{\alpha}}(t). \tag{2.9}$$

Set

$$B := \frac{\rho'_+(t)}{\rho(t)},\ A := \frac{\alpha\lambda t^{n-2}}{(n-2)!\rho^{\frac{1}{\alpha}}(t)r^{\frac{1}{\alpha}}(t)},\ \nu := u(t).$$

Using the inequality

$$-A\nu^{(\alpha+1)/\alpha} + B\nu \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{(\alpha+1)}}\frac{B^{\alpha+1}}{A^{\alpha}},\ A > 0,$$

we have

$$u'(t) \le -\rho(t)q(t)(1-p(\tau(t)))^{\alpha}(\frac{\tau^{n-1}(t)}{t^{n-1}})^{\alpha} + r(t)(\frac{\rho_{+}^{'}(t)}{\alpha+1})^{\alpha+1}(\frac{(n-2)!}{\lambda\rho(t)t^{n-2}})^{\alpha}.$$

This yields

$$\int_{t_1}^{s}\left[\rho(t)q(t)(1-p(\tau(t)))^{\alpha}(\frac{\tau^{n-1}(t)}{t^{n-1}})^{\alpha} - r(t)(\frac{\rho_{+}^{'}(t)}{\alpha+1})^{\alpha+1}(\frac{(n-2)!}{\lambda\rho(t)t^{n-2}})^{\alpha}\right]dt \le u(t_1)$$

for all large s and for every constant $\lambda \in (0,1)$, which contradicts (2.1).

Assume that case (2) holds. Integrating (1.1) from t_1 to t, we obtain

$$-r(t_1)(z^{(n-1)}(t_1))^{\alpha} + \int_{t_1}^{t} q(s)x^{\alpha}(\tau(s))ds \le 0.$$

By virtue of $z'(t) > 0$, $x(t) \ge (1-p(t))z(t)$, $\tau(s) \le s$ and (2.7)(where $n = 2$), we obtain

$$\int_{t_1}^{t} q(s)(1-p(\tau(s)))^{\alpha}(\frac{\tau(s)}{s})^{\alpha}ds \le r(t_1)(\frac{z^{(n-1)}(t_1)}{z(t_1)})^{\alpha},$$

which contradict (2.2). Integrating (1.1) from t to ∞, we conclude that

$$-r(t)(z^{(n-1)}(t))^{\alpha} + \int_{t}^{\infty} q(s)x^{\alpha}(\tau(s))ds \le 0.$$

By virtue of $z'(t) > 0$, $x(t) \ge (1-p(t))z(t)$, $\tau(s) \le s$ and (2.7)(where $n = 2$), we obtain

$$-z^{(n-1)}(t) + \frac{z(t)}{r^{\frac{1}{\alpha}}(t)}(\int_{t}^{\infty} q(s)(1-p(\tau(s)))^{\alpha}(\frac{\tau(s)}{s})^{\alpha}ds)^{\frac{1}{\alpha}} \le 0, \qquad (2.10)$$

Suppose first that $n = 4$. Integrating (2.10) from t_1 to t, we have

$$\int_{t_1}^{t} \frac{(\int_{\eta}^{\infty} q(s)(1-p(\tau(s)))^{\alpha}(\frac{\tau(s)}{s})^{\alpha}ds)^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(\eta)}d\eta \le -\frac{z^{''}(t_1)}{z(t_1)},$$

which contradicts (2.3) (where $n = 4$). Suppose now that $n \ge 6$. Integrating (2.10) from t to ∞ for a total $(n-4)$ times, we conclude that

$$-z^{'''}(t) + \frac{\int_{t}^{\infty}(\eta-t)^{(n-5)}\frac{(\int_{\eta}^{\infty} q(s)(1-p(\tau(s)))^{\alpha}(\frac{\tau(s)}{s})^{\alpha}ds)^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(\eta)}d\eta}{(n-5)!}z(t) \le 0,$$

that is

$$-z^{'''}(t) + \frac{\int_{t}^{\infty}(\eta-t)^{(n-5)}Q^{\frac{1}{\alpha}}(\eta)r^{\frac{-1}{\alpha}}(\eta)d\eta}{(n-5)!}z(t) \le 0.$$

Another integration from t_1 to t yields

$$\frac{\int_{t_1}^{\infty}(\eta-t_1)^{(n-4)}Q^{\frac{1}{\alpha}}(\eta)r^{\frac{-1}{\alpha}}(\eta)d\eta}{(n-4)!}\leq-\frac{z''(t_1)}{z(t_1)},$$

which contradicts (2.3). Integrating (2.10) from t to ∞ for a total $(n-3)$ times, we have

$$z''(t)+\frac{\int_{t}^{\infty}(\eta-t)^{(n-4)}Q^{\frac{1}{\alpha}}(\eta)r^{\frac{-1}{\alpha}}(\eta)d\eta}{(n-4)!}z(t)\leq 0. \tag{2.11}$$

Now, we define a Riccati substitution

$$\omega(t):=\delta(t)\frac{z'(t)}{z(t)},\ t\geq t_1. \tag{2.12}$$

Then $\omega(t)>0$ for $t\geq t_1$ and

$$\omega'(t)=\delta'(t)\frac{z'(t)}{z(t)}+\delta(t)\frac{z''(t)}{z(t)}-\delta(t)\frac{(z'(t))^2}{z^2(t)}.$$

It follows from (2.11) and (2.12) that

$$\begin{aligned}\omega'(t)&\leq-\delta(t)\frac{\int_{t}^{\infty}(\eta-t)^{(n-4)}Q^{\frac{1}{\alpha}}(\eta)r^{\frac{-1}{\alpha}}(\eta)d\eta}{(n-4)!}+\frac{\delta'_+(t)}{\delta(t)}\omega(t)-\frac{1}{\delta(t)}\omega^2(t) \quad (2.13)\\&\leq-\delta(t)\frac{\int_{t}^{\infty}(\eta-t)^{(n-4)}Q^{\frac{1}{\alpha}}(\eta)r^{\frac{-1}{\alpha}}(\eta)d\eta}{(n-4)!}+\frac{(\delta'_+(t))^2}{4\delta(t)}.\end{aligned}$$

This implies that

$$\int_{t_1}^{s}\left[\delta(t)\frac{\int_{t}^{\infty}(\eta-t)^{(n-4)}Q^{\frac{1}{\alpha}}(\eta)r^{\frac{-1}{\alpha}}(\eta)d\eta}{(n-4)!}-\frac{(\delta'_+(t))^2}{4\delta(t)}\right]dt\leq\omega(t_1)$$

for all large s, which contradicts (2.4). Therefore, every solution of (1.1) is oscillatory.

Let $\rho(t)=t^{n-1}$ and $\delta(t)=t$. As a consequence of Theorem 2.1 , we obtain the following oscillation criterion.

Corollary 2.5 *Let $n\geq 4$ be even and $(H_1)-(H_3)$ hold. Assume that for some constant $\lambda_0\in(0,1)$,*

$$\int^{\infty}\left[t^{n-1}q(t)(1-p(\tau(t)))^{\alpha}(\frac{\tau^{\alpha}(t)}{t^{\alpha}})^{n-1}-r(t)(\frac{(n-2)!}{\lambda_0})^{\alpha}(\frac{n-1}{\alpha+1})^{\alpha+1}t^{\alpha-n\alpha+n-2}\right]dt=\infty, \tag{2.14}$$

and either (2.2) or (2.3) or

$$\int^{\infty}\left[\frac{t}{(n-4)!}\int_{t}^{\infty}(\eta-t)^{n-4}Q^{\frac{1}{\alpha}}(\eta)r^{\frac{-1}{\alpha}}(\eta)d\eta-\frac{1}{4t}\right]dt=\infty. \tag{2.15}$$

Then (1.1) is oscillatory.

As an application of Corollary 2.1, we give the following example to illustrate our results.

Example 2.6 *Consider the equation*

$$\left[x(t) + \frac{1}{2}x(\frac{t}{2})\right]^{(4)} + \frac{a_0}{t^4}x(\frac{t}{2}) = 0, \tag{2.16}$$

where $t \geq 1$ *and* $a_0 > 0$ *is a constant. Let* $n = 4, \alpha = 1, r(t) = 1, p(t) = 1/2, q(t) = a_0/t^4$ *and* $\tau(t) = t/2$. *Then*

$$\int^{\infty} \left[t^{n-1}q(t)\left(1 - p(\tau(t))\right)^{\alpha}(\frac{\tau^{\alpha}(t)}{t^{\alpha}})^{n-1} - r(t)(\frac{(n-2)!}{\lambda_0})^{\alpha}(\frac{n-1}{\alpha+1})^{\alpha+1}t^{\alpha-n\alpha+n-2}\right] dt$$

$$= (\frac{a_0}{16} - \frac{9}{2\lambda_0})\int^{\infty}\frac{dt}{t} = \infty, \ if \ \ a_0 > \frac{72}{\lambda_0} \ \ for\ some \ \ \lambda_0 \in (0,1)$$

and

$$\int^{\infty}\left[\frac{t}{(n-4)!}\int_t^{\infty}(\eta - t)^{n-4}Q^{\frac{1}{\alpha}}(\eta)r^{\frac{-1}{\alpha}}(\eta)d\eta - \frac{1}{4t}\right] dt = (\frac{a_0}{24} - \frac{1}{4})\int^{\infty}\frac{dt}{t} = \infty, \ if\ a_0 > 6.$$

Hence by Corollary 2.1*,* (2.16) *is oscillatory if* $a_0 > 72/\lambda_0$.

The authors declaire no conflicts of interest.

ACKNOWLEDGEMENTS. This research is partly supported by the National Natural Science Foundation of Hebei Province of China (F2015201185).

References

[1] Ch.G.Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, Arch. Math., Vol. 36 (1981), 168-178.

[2] Ravi P.Agarwal, Martin Bohner, Tongxing Li, Chenghui Zhang, A new approach in the study of oscillatory behavior of even-order neutral delay differential equations, Appl. Math. Comput., Vol. 225 (2013), 787-794.

[3] R.P.Agarwal, S.R.Grace, D.O'Regan, "Oscillation Theory for Difference and Functional Differential Equations," Marcel Dekker, Kluwer Academic, Dordrecht, 2000.,

[4] Chenghui Zhang, Ravi P.Agarwal, Martin Bohner, Tongxing Li, New results for oscillatory behavior of even-order half-linear delay differential equations, Appl. Math. Lett., Vol. 26 (2013), 179-183.

[5] S.R.Grace, B.S.Lalli, Oscillation theorems for nth order nonlinear differential equations with deviating arguments, Proc. Am. Math. Soc., Vol. 90 (1984), 65-70.

[6] R.P.Agarwal, S.R.Grace, Oscillation theorems for certain functional differential equations of higher-order, Math. Comput. Model., Vol. 39 (2004), 1185-1194.

[7] R.P.Agarwal, S.R.Grace, D.O'Regan, Oscillation criteria for certain nth order differential equations with deviating arguments, J. Math. Anal. Appl., Vol. 262 (2001), 601-622.

[8] R.P.Agarwal, Martin Bohner, Tongxing Li, Chenghui Zhang, Even-order half-linear advanced differential equations: improved criteria in oscillatory and asymptotic properties, Appl. Math. Comput., Vol. 266 (2015), 481-490.

[9] C.Zhang, T.Li, B.Sun, E.Thandapani, On the oscillation of high-order half-linear delay differential equations, Appl. Math. Lett., Vol. 24 (2011), 1618-1621.

[10] Ch.G.Philos, A new ctiterion for the oscillatory and asymptotic behavior of delay differential equations, Bull. Acad. Pol. Sci. Sér. Sci. Math., Vol. 39 (1981), 61-64.

[11] I.T.Kiguradze, T.A.Chanturiya, ″Asymototic Properties of Solutions of Nonautonomous Ordinary Differential Equations,″ Kluwer Academic Publishers, Dordrecht, 1993. Translated from the 1985 Russian original.

Received: May 09, 2017