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Family of Strongly Additive Vector Measures*

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Abstract

The family of strongly additive vector measures is characterized in this paper. Firstly, the sufficient and necessary condition of a vector measure, which takes values in a completely Hausdorff topological vector space, to be strongly addtive is established. Then BTB spaces are discussed and a Diestel-Faires type result is obtained.

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1 Introduction

Vector measures have long been of interest to measure theorists, the general theory can be found in [1-4]. In recent years extensive work has been done on the additivity of vector measures ([5-14]). Olav Nygaard, Märt Põldvere ([11]) considered vector measures that take values in Banach spaces, they characterized families of vector measures of uniformly bounded variation and semivariation in terms of additivity properties, and simplified the proof of Nikodyms boundedness theorem.

Throughout this paper, $\mathcal{F}$ will be a field (i.e.,algebra) of subsets of a set and G an abelian topological group with the family $\mathcal{N}(G)$ of neighborhoods $0 \in G$. A net $(x_\alpha)_{\alpha\in(I,\leq)}$ in G is Cauchy if for every $U \in \mathcal{N}(G)$ there is an $\alpha_0 \in I$ such that $x_\alpha - x_\beta \in U$ whenever $\alpha \geq \alpha_0$ and $\beta \geq \alpha_0$. G is complete if every Cauchy net in G is convergent.

For a finitely additive measure $\mu : \mathcal{F} \to G$, the most important event is the behavior of the sequence $\{\mu(A_j)\}_{j=1}^{\infty}$ where $\{A_j\}$ is pairwise disjoint. In

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fact, $\mu : \mathcal{F} \to G$ is strongly additive if $\sum_{j=1}^{\infty} \mu(A_j)$ converges whenever $\{A_j\}$ is pairwise disjoint ([3], P.7). For Banach space X with dual X', a result in [11] says that $\mu : \mathcal{F} \to X$ is bounded variation if and only if $\sum_{j=1}^{\infty} \| \mu(A_j) \| < +\infty$ whenever $\{A_j\}$ is pairwise disjoint ([11], Cor.2), and $\mu : \mathcal{F} \to X$ is bounded, i.e., $\mu(\mathcal{F})$ is bounded ([3], P.4) if and only if whenever $\{A_j\}$ is pairwise disjoint in $\mathcal{F}$, then $\sum_{j=1}^{\infty} |x'(\mu(A_j))| < +\infty$ for each continuous linear functional $x' \in X'$ ([11], Cor.4).

In this paper, we would like to characterize the family of strongly additive vector measures in terms of additivity property. As an application, we will show a Diestel-Faires type result ([3], P.20,Theorem 2; [2]) which is also an important fact in analysis.

2 Strongly Additive Vector Measures

Definition 2.1 *For $\{x_j\} \subset G$ and $\triangle = \{j_1, j_2, \cdots\} \subseteq N$ with $j_1 < j_2 < \cdots$, let*

$$\sum_{j \in \triangle} x_j = \sum_{k=1}^{\infty} x_{j_k}$$

and $\sum_{j \in \triangle} x_j = 0$ if $\triangle = \phi$.

Ronglu Li, Hao Guo and C. Swartz ([16]) showed that every abelian topological group contains many interesting sets which are both compact and sequentially compact, they also deduced some useful facts:

Theorem A ([16, Theorem 1]). *Let Ω be a compact (resp., sequentially compact) space and G an abelian topological group. If $\{f_j\} \subset C(\Omega, G)$ is such that $\sum_{j=1}^{\infty} f_j(\omega_j)$ converges for each $\{\omega_j\} \subset \Omega$, then $\{\sum_{j=1}^{\infty} f_j(\omega_j) : \omega_j \in \Omega, \forall j \in N\}$ is compact (resp., sequentially compact).*

Theorem B ([16, Corollary 2]). *Let G be an abelian topological group and $\{x_j\} \subset G$. If $\sum_j x_j$ is subseries convergent, i.e., $\sum_{j \in \triangle} x_j$ converges for each $\triangle \subset N$, then the set $\{\sum_{j \in \triangle} x_j : \triangle \subset N\}$ is both compact and sequentially compact.*

Theorem C ([16, Theorem 2]). *Let G be an abelian topological group. Then for every countably additive $\mu : 2^N \to G$, the range $\mu(2^N) = \{\mu(A) : A \subset N\}$ is both compact and sequentially compact. Moreover, if $\sum$ is a σ -algebra of subsets of a set Ω and $\mu : \sum \to G$ is countably additive, then for every pairwise disjoint $\{A_j\} \subset \sum$, the set $\{\sum_{j \in \triangle} \mu(A_j) : \triangle \subset N\}$ is both compact and sequentially compact.*

Similar to the case of Banach spaces, we have the following simple fact.

Lemma 1. *Let G be a Hausdorff abelian topological group and $\mu : \mathcal{F} \to G$ be a vector measure. If G is complete, then the following four results are equivalent.*

(1) μ is strongly additive;

(2) If $\{A_j\}$ is pairwise disjoint in $\mathcal{F}$, then $\lim_j \mu(A_j) = 0$;
(3) If $A_1 \subseteq A_2 \subseteq \cdots$ in $\mathcal{F}$, then $\lim_j \mu(A_j)$ exists;
(4) If $A_1 \supseteq A_2 \supseteq \cdots$ in $\mathcal{F}$, then $\lim_j \mu(A_j)$ exists.

Theorem 1. *Let X be a Hausdorff topological vector space and $\mu : \mathcal{F} \to X$ be a vector measure. If X is complete, then μ is strongly additive if and only if $\{\sum_{j\in\triangle} \mu(A_j) : finite \ \ \triangle \subset N\}$ is totally bounded (i.e., precompact, [15], P.83) for every pairwise disjoint sequence $\{A_j\} \subset \mathcal{F}$.*

Proof. ($\Rightarrow$). *Let $\{A_j\}$ be pairwise disjoint in $\mathcal{F}$ and $B = \{\sum_{j\in\triangle} \mu(A_j) : finite \ \ \triangle \subset N\}$. For each $\triangle = \{j_1, j_2, \cdots\} \subseteq N$ with $j_1 < j_2 < \cdots$, we can easily get that $\{A_{j_k}\}$ is pairwise disjoint and $\sum_{j\in\triangle} \mu(A_j) = \sum_{k=1}^{\infty} \mu(A_{j_k})$ converges by the strong additivity of μ. Thus, $\sum_{j=1}^{\infty} \mu(A_j)$ is subseries convergent, i.e., $\sum_{j\in\triangle} \mu(A_j)$ converges for each $\triangle \subseteq N$.*

Let $\mathcal{S} = \{\sum_{j\in\triangle} \mu(A_j) : \triangle \subseteq N\}$. By Theorem B, $\mathcal{S}$ is both compact and sequentially compact in X. Then B is totally bounded since $B \subset \mathcal{S}$.

($\Leftarrow$). *Let $\{A_j\}$ be pairwise disjoint in $\mathcal{F}$. If $\{\sum_{j=1}^{n} \mu(A_j)\}_{n=1}^{\infty}$ is not Cauchy, then we have a balanced $U \in \mathcal{N}(X)$ and integer sequence $m_1 \le n_1 < m_2 \le n_2 < \cdots$ such that*

$$x_k = \sum_{j=m_k}^{n_k} \mu(A_j) \notin U$$

for all k.

Then pick a balanced $V \in \mathcal{N}(X)$ such that $V + V \subset U$. Since $B = \{\sum_{j\in\triangle} \mu(A_j) : finite \ \ \triangle \subset N\}$ is totally bounded, $B \subset pV$ for some $p \in N$. Then pick a balanced $W \in \mathcal{N}(X)$ for which

$$\overbrace{W + W \cdots + W}^{(p)} \subset V.$$

Since $\{x_k\} \subset B$, then $\{x_k : k \in N\}$ is totally bounded, and there is a finite $\triangle \subset N$ such that $\{x_1, x_2, x_3, \cdots\} \subset \{x_k : k \in \triangle\} + W$ ([15], P.86, Prob.6) and so there is a $k_0 \in \triangle$ such that $x_{k_0} + W$ contains infinite vectors in $\{x_k : k \in N\}$. Say that $\{x_{k_i}\}_{i=1}^{\infty} \subset x_{k_0} + W$. Then

$$\begin{aligned} \sum_{i=1}^{p} x_{k_i} &\in \overbrace{(x_{k_0} + W) + (x_{k_0} + W) + \cdots + (x_{k_0} + W)}^{(p)} \\ &= p x_{k_0} + \overbrace{W + W + \cdots + W}^{(p)} \\ &\subset p x_{k_0} + V. \end{aligned}$$

Hence $\sum_{i=1}^{p} x_{k_i} = p x_{k_0} + v$ for some $v \in V$. However, $V = -V$ and

$$x_{k_0} = \frac{1}{p}(\sum_{i=1}^{p} x_{k_i} - v) \in \frac{1}{p}(\sum_{i=1}^{p} x_{k_i} - V)$$

while

$$\begin{aligned}
\frac{1}{p}(\sum_{i=1}^{p} x_{k_i} - V) &= \frac{1}{p}(\sum_{i=1}^{p} x_{k_i} + V) \\
&= \frac{1}{p}[\sum_{i=1}^{p} \sum_{j=m_{k_i}}^{n_{k_i}} \mu(A_j) + V] \\
&\subset \frac{1}{p}(B + V) \\
&\subset \frac{1}{p}(pV + V) \\
&= V + \frac{1}{p}V \\
&\subset V + V \subset U.
\end{aligned}$$

This contradicts that $x_k \notin U$ for all k and so $\{\sum_{j=1}^{n} \mu(A_j)\}_{n=1}^{\infty}$ is Cauchy in X. Since X is complete then $\sum_{j=1}^{\infty} \mu(A_j) = \lim_n \sum_{j=1}^{n} \mu(A_j)$ exists.

3 Diestel-Faires Type Result

A topological vector space X is called a BTB space if every bounded set in X is totally bounded ([15], P.85). Many important spaces are BTB spaces, e.g., $R^n, C^n, R^{\mathcal{N}}, C^{\mathcal{N}}(=\omega)$, the space $\mathcal{D}$ of test functions, semi-Montel spaces ([15], P.90), etc. In fact, we have many BTB spaces as follows.

Lemma 3.1 *Let X be a complete Hausdorff BTB space and X^Ω all the mappings from Ω to X. For every $\Omega \neq \phi$, let $\sigma\Omega$ be the topology for X^Ω such that $f_\alpha \to f$ in $(X^\Omega, \sigma\Omega)$ if and only if $f_\alpha(\omega) \to f(\omega), \forall \omega \in \Omega$. Then $(X^\Omega, \sigma\Omega)$ is a complete Hausdorff BTB space.*

Proof. *Each $\omega \in \Omega$ gives a function $\omega : X^\Omega \to X$ such that $\omega(f) = f(\omega)$, $\forall f \in X^\Omega$. Letting $\omega^{-1}(\phi) = \phi$ and denoting $\sigma\omega = \{\omega^{-1}(G) : G$ is open in $X\}$ be a topology for X^Ω and $\sigma\Omega = sup\{\sigma\omega : \omega \in \Omega\}$ ([15], P.11).*

If $f, g \in X^\Omega, f \neq g$, then $f(\omega_0) \neq g(\omega_0)$ for some $\omega_0 \in \Omega$ and so $[f(\omega_0) + U] \cap [g(\omega_0) + U] = \phi$ for some open $U \in \mathcal{N}(X)$. Then $\omega_0^{-1}[f(\omega_0) + U] \cap \omega_0^{-1}[g(\omega_0) + U] = \phi$ and so $(X^\Omega, \sigma\Omega)$ is Hausdorff.

Let $(f_\alpha)_{\alpha \in (I, \leq)}$ be a Cauchy net in $(X^\Omega, \sigma\Omega)$. Then $(f_\alpha)_{\alpha \in I}$ is Cauchy in $(X^\Omega, \sigma\omega)$ for each $\omega \in \Omega$ ([15], P76, Prob.4). Fix an $\omega \in \Omega$. For every open $V \in \mathcal{N}(X), \omega^{-1}(V) = \{f \in X^\Omega : f(\omega) = \omega(f) \in V\}$ is an open neighborhood of $0 \in (X^\omega, \sigma\omega)$ and so there is $\alpha_0 \in I$ such that $f_\alpha - f_\beta \in \omega^{-1}(V)$ for all $\alpha, \beta \geq \alpha_0$, i.e., $f_\alpha(\omega) - f_\beta(\omega) = (f_\alpha - f_\beta)(\omega) = \omega(f_\alpha - f_\beta) \in V$ for all $\alpha, \beta \geq \alpha_0$. Then $(f_\alpha(\omega))_{\alpha \in I}$ is Cauchy in X and so $\lim_\alpha f_\alpha(\omega) = f(\omega)$ exists.

Thus, we have a $f \in X^{\Omega}$ *such that* $f_\alpha \to f$ *in* $(X^{\Omega}, \sigma\Omega)$*, i.e.,* $(X^{\Omega}, \sigma\Omega)$ *is complete.*

Let $\mathcal{S}$ *be a bounded set in* $(X^{\Omega}, \sigma\Omega)$ *and* $\overline{\mathcal{S}}^{\sigma\Omega}$ *be the closure of* $\mathcal{S}$ *in* $(X^{\Omega}, \sigma\Omega)$ *which is also bounded. If* $\{f_n\} \subset \overline{\mathcal{S}}^{\sigma\Omega}$*, then* $\frac{1}{n}f_n \overset{\sigma\Omega}{\to} 0$ *and so* $\frac{1}{n}f_n(\omega) \to 0$ *for all* $\omega \in \Omega$*, i.e.,* $\{f(\omega) : f \in \overline{\mathcal{S}}^{\sigma\Omega}\}$ *is bounded in* X *for each* $\omega \in \Omega$*. Since* X *is a complete* BTB *space,* $\overline{\{f(\omega) : f \in \overline{\mathcal{S}}^{\sigma\Omega}\}}$ *is totally bounded and complete in* X *for each* $\omega \in \Omega$*, i.e.,* $\overline{\{f(\omega) : f \in \overline{\mathcal{S}}^{\sigma\Omega}\}}$ *is compact for each* $\omega \in \Omega$ *([15], P.88, Th.7). Then* $\overline{\mathcal{S}}^{\sigma\Omega}$ *is compact in* $(X^{\Omega}, \sigma\Omega)$ *([17], P218, Th.1) and so* $\mathcal{S}$ *is totally bounded in* $(X^{\Omega}, \sigma\Omega)$*.*

A very nice Diestel-Faires theorem shows that a Banach space X *contains no copy of* c_0 *if and only if every bounded measure* $\mu : \mathcal{F} \to X$ *is strongly additive ([3], P.20, Th.2; [2]). But Lemma 2 shows that the family of* BTB *spaces includes many of non-metrizable spaces and so we have a nice fact as follows.*

Theorem 3.2 *Let* X *be a complete Hausdorff* BTB *space. If* $\mu : \mathcal{F} \to X$ *is a bounded measure, i.e.,* $\mu(\mathcal{F}) = \{\mu(A) : A \in \mathcal{F}\}$ *is bounded in* X *, then* μ *is strongly additive and for every pairwise disjoint sequence* $\{A_j\} \subset F$*,* $\{\sum_{j\in\triangle} \mu(A_j) : \triangle \subseteq N\}$ *is both compact and sequentially compact.*

Proof. *Let* $\{A_j\}$ *be pairwise disjoint in* $\mathcal{F}$*. Then*

$$\mathcal{K} = \{\sum_{j\in\triangle} \mu(A_j) : finite \ \ \triangle \subseteq \mathcal{N}\} = \{\mu(\cup_{j\in\triangle} A_j) : finite \ \ \triangle \subseteq \mathcal{N}\} \subset \mu(\mathcal{F})$$

is bounded in X *and so* $\mathcal{K}$ *is totally bounded. By Theorem 1,* μ *is strongly additive.*

As in the proof of Theorem A and Theorem B implies that $\{\sum_{j\in\triangle} \mu(A_j) : \triangle \subseteq \mathcal{N}\}$ *is both compact and sequentially compact whenever* $\{A_j\}$ *is pairwise disjoint in* $\mathcal{F}$*.* $\square$

Theorem C says that if $\mu : 2^{\mathcal{N}} \to G$ *is countably additive, then the range* $\mu(2^{\mathcal{N}}) = \{\mu(A) : A \subseteq \mathcal{N}\}$ *is both compact and sequentially compact . For strongly additive measures we also have a similar result as follows.*

Corollary 3.3 *Let* X *be a Hausdorff topological vector space.If* X *is complete and* $\mu : 2^{\mathcal{N}} \to X$ *is a strongly additive measure, then* $\{\sum_{j\in A} \mu(j) : A \subseteq \mathcal{N}\}$ *is both compact and sequentially compact.*

Proof. *If* $i \neq j$ *in* $\mathcal{N}$*, then* $\{i\} \cap \{j\} = \phi$ *and* $A = \cup_{j\in A}\{j\}$ *whenever* $A \subseteq \mathcal{N}$*. Then* $\sum_{j\in A} \mu(\{j\})$ *converges for each* $A \subseteq \mathcal{N}$*. As in the proof of Theorem 1, the desired result follows from Theorem B.*

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$\mu : \mathcal{F} \to X$, where X is a complete Hausdorff topological vector space, strongly additive if and only if $\{\sum_{j\in\triangle} \mu(A_j) : finite \ \ \triangle \subset N\}$ is totally bounded for every pairwise disjoint sequence $\{A_j\} \subset \mathcal{F}$.

We also discussed the BTB space, and established a Diestel-Faires type theorem: If X be a complete Hausdorff BTB space. and $\mu : \mathcal{F} \to X$ is a bounded measure, then μ is strongly additive and for every pairwise disjoint sequence $\{A_j\} \subset F$, $\{\sum_{j\in\triangle} \mu(A_j) : \triangle \subseteq N\}$ is both compact and sequentially compact.

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