Mathematica Aeterna, Vol. 7, 2017, no. 3, 295 - 300

Non-Hermitian Hamiltonian in $PT-$symmetric Quantum System*

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Abstract

The specific forms of $PT-$symmetric Hamiltonians in 2×2 quantum system are studied in this paper. Depending on the relationship between the non-Hermitian and Hermitian matrices and the special property of the Hamiltonian satisfying PT symmetry, the specific forms of the non-Hermitian but PT symmetric Hamiltonian are presented, and thus a general method for discussing such problems in higher dimension systems is established.

Mathematics Subject Classification: Quantum theory

Keywords: $PT-$symmetry, Hamiltonian, Hermitian matrix

1 Introduction

In classical quantum system, the observables are represented by Hermitian operators, however, non-Hermitian observables also play vital roles in physics[1-10], which is contrary to classical quantum mechanics. In order to solve the complex situation, Bender C. M. et al[1]. put forward PT symmetric quantum theory in 1998, which pointed out that non-Hermitian Hamiltonians had real eigenvalues provided they respect unbroken PT symmetry. PT symmetry is refers to the parity-time symmetry, where P and T stand for parity and time reversal respectively.

In quantum mechanics, $\hat{x}$ and $\hat{p}$ stand for coordinate operator and momentum operator, respectively. Their algorithm is as follows[1]:

$$(\hat{x}f)(x,t) = xf(x,t), \qquad (\hat{p}f)(x,t) = -i\frac{\partial}{\partial x}f(x,t).$$

*Supported by Natural Science Foundation of China (11361065,11761073)

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If an operator P satisfies the following equality

$$P\hat{x}P = -\hat{x}, \qquad P\hat{p}P = -\hat{p}, \tag{1}$$

Then P is called parity operator (or space inversion operator)[1], in short operator$P$. Obviously, it is a linear operator. If operator T satisfies

$$T\hat{x}T = -\hat{x}, \qquad T\hat{p}T = -\hat{p}, \qquad TiT = -i, \tag{2}$$

where $i = \sqrt{-1}$, then T is called time reversal operator[12], in short operator T. Obviously, it is a conjugate-linear operator.

If H is a $n \times n$ matrix satisfying

$$H = H^{PT}, \tag{3}$$

where $H^{PT} = (PT)H(PT)$, then we say that H is $PT-$symmetric.

By the definition of operator T, time reversion operator is anti-linear, namely conjugate linear, therefore, it can be divided into the following two categories in general:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \quad or \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\bar{x} \\ -\bar{y} \end{pmatrix}, \tag{4}$$

where $\overline{x}$ stands for the conjugate of x. We denote the above two kinds of operator T as T_1 and T_2. Obviously,$T_1^2 = T_2^2 = I$ (unit operator).

This paper mainly discusses the specific forms of $PT-$symmetric Hamiltonians in 2×2 and 3×3 quantum system. Using the relationship between the non-Hermitian matrix and Hermitian matrix, and the special property of the Hamiltonian satisfying PT symmetry, it present the specific form of the non - Hermitian Hamiltonian satisfying PT symmetric conditions is obtained, and thus give a general method for discussing such problems in higher dimension systems.

2 Preliminary Notes

We begin with the relationship between non-Hermitian matrix and Hermitian matrix[11], and the special property of the Hamiltonian satisfying PT symmetry[12].

Lemma 2.1 [11] *Every non-Hermitian matrix N can be expressed by two Hermitian matrices as follows:*

$$N = \frac{1}{2}(H_1 + iH_2), \tag{5}$$

where $H_1, H_2$are both Hermitian matrix, which are not zero matrices. Corresponding, the transpose conjugate matrix $N^\dagger$ of N has the following expression

$$N^\dagger = \frac{1}{2}(H_1 - iH_2), \tag{6}$$

If H_1, H_2 are both 2×2 Hermitian matrices, then they must be the following forms:

$$H_1 = \begin{pmatrix} a_1 & b_1 \\ \overline{b_1} & c_1 \end{pmatrix}; \quad H_2 = \begin{pmatrix} a_2 & b_2 \\ \overline{b_2} & c_2 \end{pmatrix}, \tag{7}$$

where $a_1, a_2, c_1, c_2 \in R, \quad b_1, b_2 \in C$. Note that H_2 is not a zero matrix, i.e.a_2, b_2, c_2 do not equal zero simultaneously.

It is from (5) that any 2×2 non-Hermitian matrix H_N can be represent as follows:

$$H_N = \begin{pmatrix} \frac{a_1+ia_2}{2} & \frac{b_1+ib_2}{2} \\ \frac{\overline{b_1}+i\overline{b_2}}{2} & \frac{c_1+ic_2}{2} \end{pmatrix}, \tag{8}$$

where $a_1, a_2, c_1, c_2 \in R, \quad b_1, b_2 \in C$. and a_2, b_2, c_2 do not equal zero simultaneously.

Lemma 2.2 [12] *Assuming that H is a Hamiltonian of 2×2 quantum system, if H meets PT symmetry, no matter $T = T_1$ or $T = T_2$, for same operator P, they all have $P\overline{H} = HP$.*

Lemma 2.3 [12] *In finite dimensional space, any operator P, which is commutate to operator T, is a real matrix.*

According to the above lemmas, we have established the forms of operator P in 2×2 quantum system[12], in this paper we choose the following form of operator P:

$$P = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}, \alpha \in R. \tag{9}$$

3 Main Results

In this section, let operator T be complex conjugate operator P, operator take form (9). We then present the concrete form of non-Hermitian Hamiltonian H_N which satisfies the PT symmetry in 2×2 quantum system.

It follows from Lemma 1 that any 2×2 non-Hermitian matrix H_N can be represented as (8). If H_N satisfies PT symmetry, then we can calculate the following two quantities:

$$P\overline{H_N} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \begin{pmatrix} \frac{a_1-ia_2}{2} & \frac{\overline{b_1}-i\overline{b_2}}{2} \\ \frac{b_1-ib_2}{2} & \frac{c_1-ic_2}{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \cos\alpha(a_1 - ia_2) + \sin\alpha(b_1 - ib_2) & \cos\alpha(\overline{b_1} - i\overline{b_2}) + \sin\alpha(c_1 - ic_2) \\ \sin\alpha(a_1 - ia_2) - \cos\alpha(b_1 - ib_2) & \sin\alpha(\overline{b_1} - i\overline{b_2}) - \cos\alpha(c_1 - ic_2) \end{pmatrix}, \tag{10}$$

$$H_N P = \begin{pmatrix} \frac{a_1+ia_2}{2} & \frac{b_1+ib_2}{2} \\ \frac{\overline{b_1}+i\overline{b_2}}{2} & \frac{c_1+ic_2}{2} \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \cos\alpha(a_1 + ia_2) + \sin\alpha(b_1 + ib_2) & \sin\alpha(a_1 + ia_2) - \cos\alpha(b_1 + ib_2) \\ \cos\alpha(\overline{b_1} + i\overline{b_2}) + \cos\alpha(c_1 + ic_2) & \sin\alpha(\overline{b_1} + i\overline{b_2}) - \cos\alpha(c_1 + ic_2) \end{pmatrix}. \tag{11}$$

Note that $P\overline{H_N} = H_N P$ by Lemma 2, so from (10) and (11) we have

$$\begin{cases} a_2\cos\alpha + b_2\sin\alpha = 0 \\ (b_1 + \overline{b_1})\cos\alpha = (a_1 - c_1)\sin\alpha \\ (b_2 - \overline{b_2})\cos\alpha = (a_2 + c_2)\sin\alpha \\ c_2\cos\alpha = \overline{b_2}\sin\alpha \end{cases} \tag{12}$$

where $a_1, a_2, c_1, c_2 \in R$, $\quad b_1, b_2 \in C$. and a_2, b_2, c_2 do not equal zero simultaneously. We can easily get that $b_2 \in R$ from the fourth equality in (12).

In order to fully ensure the relationship between various parameters in (12), and further specific the forms of H_N, we analyze (12) in two cases: I,$\cos\alpha = 0$; II,$\cos\alpha \neq 0$.

I, If $\cos\alpha = 0$, then $\sin\alpha \neq 0$, so

$$b_2 = 0, \quad a_1 = c_1, \quad a_2 = -c_2,$$

, then we have $0 \neq a_2 \in R, b_1 \in C$, thus

$$H_N = \frac{1}{2}\begin{pmatrix} a_1 + ia_2 & b_1 \\ \overline{b_1} & a_1 - ia_2 \end{pmatrix}, 0 \neq a_1, a_2 \in R, b_1 \in C. \tag{13}$$

For example, we can take H_N as follows,

$$\begin{pmatrix} 1+i & i \\ -i & 1-i \end{pmatrix}. \tag{14}$$

II, If $\cos\alpha \neq 0$, then (12) can be changed into

$$\begin{cases} a_1 + b_2\tan\alpha = 0 \\ (b_1 + \overline{b_1}) = (a_1 - c_1)\tan\alpha \\ (b_2 - \overline{b_2}) = (a_2 + c_2)\tan\alpha \\ c_2 = \overline{b_2}\tan\alpha \end{cases} \tag{15}$$

1) If $\tan\alpha = 0$, then

$$b_1 \in R, \quad 0 \neq b_2 \in R, \quad a_1, c_1 \in R, \quad a_2 = c_2 = 0$$

hence

$$H_N = \frac{1}{2}\begin{pmatrix} a_1 & b_1 + ib_2 \\ -b_1 + ib_2 & c_1 \end{pmatrix}, b_1 \in R, 0 \neq b_2 \in R, a_1, c_1 \in R, \quad (16)$$

For example, we can take H_N as follows .

$$\begin{pmatrix} 2 & 2+i \\ -2+i & 1 \end{pmatrix}. \quad (17)$$

2)If $\tan\alpha \neq 0$, then (12) can be changed into

$$\begin{cases} -c_2 = a_2 \\ b_2 = -\frac{1}{\tan\alpha} a_2 \\ (b_1 + \overline{b_1}) = (a_1 - c_1)\tan\alpha \end{cases} \quad (18)$$

so

$$H_N = \frac{1}{2}\begin{pmatrix} a_1 + ia_2 & b_1 - i\frac{a_2}{\tan\alpha} \\ \overline{b_1} - i\frac{a_2}{\tan\alpha} & a_2 - i\frac{\overline{b_1}+b_1}{\tan\alpha} - ia_2 \end{pmatrix}, 0 \neq a_1, a_2 \in R, b_1 \in C, \quad (19)$$

For example, we may take H_N as follows

$$\begin{pmatrix} 1+i & 0 \\ -2i & 1-i \end{pmatrix}. \quad (20)$$

ACKNOWLEDGEMENTS.

This paper mainly discussed the concrete form of non-Hermitian Hamiltonian satisfying PT symmetry condition in 2×2 quantum system. Depending on the relationship between the non-Hermitian and Hermitian matrices and the special property of the Hamiltonian satisfying PT symmetry, $P\overline{H_N} = H_N P$, it analyzed the specific forms of the non-Hermitian under different conditions. It also presented a general method for discussing such problems in higher dimension systems.

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Received: August 30, 2017