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SOME PROPERTIES OF FUNCTIONS FROM GENERALIZED SOBOLEV-MORREY TYPE SPACES

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Abstract

TIn this paper, we introduce a generalized Sobolev-Morrey type spaces. Furthermore, the Sobolev type embedding theorems in this spaces is established.

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1 Introduction and preliminary notes

In this paper we first intordUce a generalized Sobolev-Morrey type spaces

$$\bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G) \tag{1}$$

where $G \subset R^n$; $1 \leq p^i < \infty$; $l^i \in N_0^n$, $(i = 0, 1, \ldots, n)$, $l_j^0 \geq 0$ are entire $(j = 1, \ldots, n)$, $l_j^i \geq 0$ are entire $(i \neq j = 1, \ldots, n)$, $l_i^i > 0$ are entire $(i = 1, \ldots, n)$; $\beta \in [0, 1]^n$; $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$, $\varphi_j(t) > 0 (t > 0)$ by Lebesgue measurable functions; $\lim\limits_{t\to+0} \varphi_j(t) = 0$, and $\lim\limits_{t\to+\infty} \varphi_j(t) = \infty$. We denote by A the set of all vector functions. Further, using the integral representation method we study some differential properties of functions, defined in n-dimensional domains satisfying "flexible φ- horn" condition.

Definition 1.1 *The spaces* $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$ *under consideration consists of the set locally summable on G functions f having on G the generalized mixed derivatives* $D^{l^i}f$ *with the finite norm*

$$\|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G)} = \sum_{i=0}^{n} \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta;G}, \tag{2}$$

where

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{x\in G,\, t>0} \left(|\varphi([t]_1)|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)} \right). \tag{3}$$

$|\varphi([t]_1)|^{-\beta} = \prod\limits_{j=1}^{n} (\varphi_j([t]_1))^{-\beta_j}$,$\beta_j \in (0,1]$, $(j = 1, 2, \dots, n)$ *and* $[t]_1 = \min\{1, t\}$. *For any* $x \in R^n$

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), j = 1, \dots, n \right\}.$$

Note that the spaces with parameters introduced and studied in [3], [6], [8], [9], [10], [12], [13], [15], [16] and others.

Let for any $t > 0$ $|\varphi([t]_1)| \le C$, where $C > 0$ is positive constant. Then the embeddings $L_{p,\varphi,\beta}(G) \hookrightarrow L_p(G)$, $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G) \hookrightarrow \bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i}(G)$ hold i.e.

$$\|f\|_{p;G} \le C\|f\|_{p,\varphi,\beta;G},$$

and

$$\|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i}(G)} \le C\|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G)}$$

Note that the spaces $L_{p,\varphi,\beta}(G)$ and $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$ are Banach spaces. The compleness of the these spaces automatically implies from completeness of $L_p(G)$ and $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i}(G)$.

The space $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i}(G)$, when in the case $\beta_j = 0$ $(j = 1, \dots, n)$ coincides with the $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i}(G)$ introduced and studied in [7]. in the case $l^0 = (0, \dots, 0)$, $l^i = (0, \dots, 0, l_i, 0, \dots, 0)$, $p^i = p$, $(i = 0, 1, \dots, n)$ coincides with the spaces $W^l_{p,\varphi,\beta}(G)$ introduced and studied in [11] . The spaces of such type with different norms were introduced and studied in [1], [4], [5], [14] and others.

Let G be a bounded domain and $p \le q$, $\varphi(t) \le \psi(t)$ $(t > 0)$; $\exists c > 0$, $\forall t \in (0, 1)$, $|\psi(t)|^{\beta_1} \le C_1|\varphi(t)|^{\beta}$, then $L_{q,\psi,\beta_1}(G) \hookrightarrow L_{p,\varphi,\beta}(G)$,i.e. exactly there exists $C > 0$ such that

$$|f\|_{p,\varphi,\beta;G} \le C_2\|f\|_{q,\psi,\beta_1;G}. \tag{4}$$

Definition 1.2 *The open set $G \subset R^n$ is said to be an open set with condition of type flexible φ-horn if for some $\theta \in (0,1]^n$, $T \in (0,\infty)$ for any $x \in G$ there exists the vector-function*

$$\rho(\varphi(t), x) = (\rho_1(\varphi_1(t), x), \ldots, \rho_n(\varphi_n(t), x)), \quad 0 \le t \le T$$

with the following properties:

1) for all $j \in e_n$ $\rho(\varphi_j(t), x)$ is absolutely continuous on $[0,T]$, $\left|\rho_j'(\varphi_j(t), x)\right| \le 1$ for almost all $t \in [0,T]$,

2) $\rho_j(0,x) = 0$, $x + V(x,\theta) = \bigcup\limits_{0 \le t \le T} [\rho(\varphi(t), x) + \varphi(t)\theta I] \subset G$.

In particular, $\varphi(t) = t^\lambda$ $\left(t^\lambda = \left(t^{\lambda_1}, t^{\lambda_2}, ..., t^{\lambda_n}\right)\right)$ and $\theta_j = \theta^{\lambda_j}$ $(j = 1, \ldots, n)$ is the set $x + V(x, \lambda, \theta)$ will called the flexible λ-horn introduced in [2].

Assuming that $\varphi_j(t)$ $(j = 1, 2, \ldots, n)$ are also differentiable on $[0,T]$, we can show that for $f \in \bigcap\limits_{i=0}^{n} L_{p^i}^{<l^i>}(G)$ determined in n- dimensional domains, satisfying the condition of flexible φ-horn, it holds the following integral representation $(\forall x \in U \subset G)$

$$D^\nu f(x) = f_{\varphi(T)}^{(\nu)}(x) +$$

$$\sum_{i=0}^{n} \int_0^T \int_{R^n} M_i^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \rho'(\varphi(t), x)\right) \times$$

$$D^{l^i} f(x+y) \prod_{j=1}^{n} (\varphi_j(t))^{l_j^i - \nu_j - 1} \prod_{j \in e_{l^i}} \frac{\varphi_j'(t)}{\varphi_j(t)} dt dy, \tag{5}$$

$$f_{\varphi(T)}^{(\nu)}(x) = (-1)^{|\nu| + |l^0|} \prod_{j=1}^{n} (\varphi_j(T))^{l_j^0 - \nu_j - 1} \times$$

$$\times \int_{R^n} D^{l^0} f(x+y)\, \Omega^{(\nu)}\left(\frac{y}{\varphi(T)}, \frac{\rho(\varphi(T), x)}{\varphi(T)}\right) \tag{6}$$

where e_{l^i} the set of indices of nonzero components of the vector l^i. Let $L(\cdot, y, z) \in C_0^\infty(R^n)$ be such that

$$S(L) \subset I_{\varphi(T)} = \left\{y : |y_j| < \frac{1}{2}\varphi_j(T), \quad j = 1, 2, ..., n\right\}.$$

Assume that for any $0 < T \le 1$ and

$$V = \bigcup_{0 < t \le T} \left\{y : \frac{y}{\varphi(t)} \in S(L)\right\}.$$

It is clear that $V \subset I_{\varphi(T)}$ and suppose that $U + V \subset G$.

Lemma 1.3 *Let $1 \le p^i \le p \le r \le \infty$; $0 < \eta$, $t < T \le 1$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\nu_j \ge 0$ be entire, $(j = 1, \ldots, n)$; $\Psi \in L_{p^i,\varphi,\beta}(G)$ and*

$$Z_\eta^i(x) = \int_0^\eta L\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \rho'(\varphi(t), x)\right)\Phi(x+y) \times$$
$$\prod_{j=1}^n (\varphi_j(t))^{l_j^i - \nu_j - 1} \prod_{j \in e_{l^i}} \frac{\varphi_j'(t)}{\varphi_j(t)} dt dy, \tag{7}$$

$$Z_{\eta,T}^i(x) = \int_0^\eta L\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \rho'(\varphi(t), x)\right)\Phi(x+y) \times$$
$$\prod_{j=1}^n (\varphi_j(t))^{l_j^i - \nu_j - 1} \prod_{j \in e_{l^i}} \frac{\varphi_j'(t)}{\varphi_j(t)} dt dy, \tag{8}$$

and let

$$Q_T^i = \int_0^T \prod_{j=1}^n (\varphi_j(t))^{l_j^i - \nu_j - 1 - \left(1 - \beta_j p^i\right)\left(\frac{1}{p^i} - \frac{1}{p}\right)} \prod_{j \in e_{l^i}} \frac{\varphi_j'(t)}{\varphi_j(t)} dt < \infty.$$

Then for any $\overline{x} \in U$ the following inequalities are true

$$\sup_{\overline{x} \in U} \left\| Z_\eta^i \right\|_{p, U_{\psi(\xi)}(\overline{x})} \le C_1 \|\Phi\|_{p^i,\varphi,\beta;G} \times$$
$$|Q_\eta^i| \prod_{j=1}^n (\psi_j([\xi]_1))^{\beta_j \frac{p^i}{p}}, \tag{9}$$

$$\sup_{\overline{x} \in U} \left\| Z_{\eta,T}^i \right\|_{p, U_{\psi(\xi)}(\overline{x})} \le C_2 \|\Phi\|_{p^i,\varphi,\beta;G} \times$$
$$|Q_{\eta,T}^i| \prod_{j=1}^n (\psi_j([\xi]_1))^{\beta_j \frac{p^i}{p}}, \tag{10}$$

where $U_{\psi(\xi)}(\overline{x}) = \left\{x : |x_j - \overline{x}_j| < \frac{1}{2}\psi_j(\xi), j = 1, 2, ..., n\right\}$ and $\psi \in A$, C_1, C_2 are the constants independent of φ, ξ, η and T.

Proof. Applying sequentially the generalized Minkowskii inequality for any $\overline{x} \in U$

$$\left\| Z_\eta^i \right\|_{p, U_{\psi(\xi)}(\overline{x})} \le \int_0^\eta \|R(\cdot, t)\|_{p, U_{\psi(\xi)}(\overline{x})} \times$$
$$\prod_{j=1}^n (\varphi_j(T))^{l_j^i - \nu_j - 1} \prod_{j = e_{l^i}} \frac{\varphi_j'(t)}{\varphi_j(t)} dt, \tag{11}$$

where

$$R(x,t) = \int\limits_{R^n} L\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \rho'(\varphi(t),x)\right)\Phi(x+y)\,dy. \tag{12}$$

From the Hölder inequality ($p \le r$) we have

$$\|R(\cdot,t)\|_{p,U_{\psi(\xi)}(\overline{x})} \le \|R(\cdot,t)\|_{r,U_{\psi(\xi)}(\overline{x})} \prod_{j=1}^{n} (\psi_j(\xi))^{\frac{1}{p}-\frac{1}{r}}. \tag{13}$$

Now estimate the norm $\|R(\cdot,t)\|_{p,U_{\psi(\xi)}(\overline{x})}$.

Let χ be a characteristic function of the set $S(L)$. Assume that $|L(x,y,z)| \le C|L_1(x)|$, for all $(y,z) \in R^n \times R^n$, and $L_1 \in C_0^\infty(R^n)$. Noting that $1 \le p^i \le r \le \infty$, $s \le r$ represent the integrand function (12) in the form

$$|L\Phi| = \left(|\Phi|^{p^i}|L|^s\right)^{\frac{1}{r}} \left(|\Phi|^{p^i}\chi\right)^{\frac{1}{p^i}-\frac{1}{r}} (|L|^s)^{\frac{1}{s}-\frac{1}{r}}$$

and apply for $|R|$ the Hölder inequality $\left(\frac{1}{r} + \left(\frac{1}{p^i} - \frac{1}{r}\right) + \left(\frac{1}{s} - \frac{1}{r}\right) = 1\right)$, we obtain

$$\|R(\cdot,t)\|_{r,U_{\psi(\xi)}(\overline{x})} \le$$

$$\sup_{x\in U_{\psi(\xi)}(\overline{x})} \left(\int\limits_{R^n} |\Phi(x+y)|^{p^i} \chi\left(\frac{y}{\varphi(t)}\right) dy\right)^{\frac{1}{p^i}-\frac{1}{r}} \times$$

$$\sup_{y\in V} \left(\int\limits_{U_{\psi(\xi)}(\overline{x})} |\Phi(x+y)|^{p^i} dx\right)^{\frac{1}{r}}$$

$$\left(\int\limits_{R^n} \left|L_1\left(\frac{y}{\varphi(t)}\right)\right|^s dy\right)^{1/s}. \tag{14}$$

For any $x \in U$ we have

$$\int\limits_{R^n} |\Phi(x+y)|^{p^i} \chi\left(\frac{y}{\varphi(t)}\right) dy \le \int\limits_{(U+V)_{\varphi(t)}(\overline{x})} |\Phi(y)|^{p^i} dy$$

$$\le \int\limits_{G_{\varphi(t)}(\overline{x})} |\Phi(y)|^{p^i} dy \le \|\Phi\|^{p^i}_{p^i,\varphi,\beta;G} \prod_{j=1}^{n} (\varphi_j(t))^{\beta_j p^i}. \tag{15}$$

For $y \in V$

$$\int\limits_{U_{\psi(\xi)}(\overline{x})} |\Phi(x+y)|^{p^i} dx \le \int\limits_{U_{\psi(\xi)}(\overline{x}+y)} |\Phi(x)|^{p^i} dx \le$$

$$\leq \|\Phi\|_{p^i,\psi,\beta;U}^{p^i} \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j p^i} \leq$$

$$\leq C_1 \|\Phi\|_{p^i,\varphi,\beta;G}^{p^i} \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j p^i} \quad (U_{\psi(\xi)}(x) \subset G_{\varphi(\xi)}(x)), \tag{16}$$

$$\int\limits_{R^n} \left| L\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \rho'(\varphi(t), x)\right)\right|^s dy = \|L\|_s^s \prod_{j=1}^{n} \left(\varphi_j(t)\right). \tag{17}$$

From inequalities (14)- (17) it follows that

$$\|R(\cdot,t)\|_{r,U_{\psi(\xi)}(\overline{x})} \leq \|L\|_s \|\Phi\|_{p^i,\varphi,\beta;G} \times$$

$$\times \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{\frac{1}{s}+\beta_j p^i\left(\frac{1}{p^i}-\frac{1}{r}\right)} \prod_{j=1}^{n} \left(\psi_j\left([\xi_j]_1\right)\right)^{\frac{\beta_j p^i}{r}}. \tag{18}$$

Inequalities in (11),(13) and (18) for $(r = p)$ and for any $\overline{x} \in U$ reduce to the estimation

$$\left\|Z_\eta^i\right\|_{p,U_{\psi(\xi)}(\overline{x})} \leq C_1 \|\Phi\|_{p^i,\varphi,\beta;G} |Q_\eta^i| \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j \frac{p^i}{p}}. \tag{19}$$

In the case $Q_{\eta,T}^i < \infty$ inequality (10) is proved in the same way

$$\left\|Z_{\eta,T}^i\right\|_{p,U_{\psi(\xi)}(\overline{x})} \leq C_2 \|\Phi\|_{p^i,\varphi,\beta;G} |Q_{\eta,T}^i| \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j \frac{p^i}{p}}. \tag{20}$$

From inequality (18) and (20) we get inequality $(\forall \overline{x} \in U)$

$$\sup_{\overline{x}\in U} \|R\|_{p,U_{\psi(\xi)}(\overline{x})} \leq C_3 \|\Phi\|_{p^i,\varphi,\beta;G} \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j \frac{p^i}{p}}, \tag{21}$$

$$\sup_{\overline{x}\in U} \left\|Z_\eta^i\right\|_{p,U_{\psi(\xi)}(\overline{x})} \leq C_4 \|\Phi\|_{p^i,\varphi,\beta;G} \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j \frac{p^i}{p}} \tag{22}$$

$$\sup_{\overline{x}\in U} \left\|Z_{\eta,T}^i\right\|_{p,U_{\psi(\xi)}(\overline{x})} \leq C_5 \|\Phi\|_{p^i,\varphi,\beta;G} \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j \frac{p^i}{p}} \tag{23}$$

From last inequalities it follows that

$$\|R\|_{p,\psi,\beta_1;U} \leq C_1' \|\Phi\|_{p^i,\varphi,\beta;G}. \tag{24}$$

$$\left\|Z_\eta^i\right\|_{p,\psi,\beta_1;U} \leq C_2' \|\Phi\|_{p^i,\varphi,\beta;G}. \tag{25}$$

$$\left\|Z_{\eta,T}^i\right\|_{p,\psi,\beta_1;U} \leq C_3' \|\Phi\|_{p^i,\varphi,\beta;G}. \tag{26}$$

C_1', C_2' and C_3' are the constants independent of Φ.

This complete the proof of Lemma 1.3.

2 Main results

Prove two theorems on the properties of the functions from the space $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$.

Theorem 2.1 *Let $G \subset R^n$ satisfy the condition of flexible φ-horn, $1 \le p^i \le p \le \infty$ $(i = 0,1,\dots,n)$, $\nu = (\nu_1,\nu_2,\dots,\nu_n)$, $\nu_j \ge 0$ be entire $(j = 1,\dots,n)$, and suppose that $\nu_j \ge l_j^0$; $\nu_j \ge l_j^i$ $(j \ne i = 1,\dots,n)$, $Q_T^i < \infty (i = 1,\dots,n)$ and let $f \in \bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$. Then the following embeddings hold*

$$D^\nu : \bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G) \to L_{p,\psi,\beta^1}(G)$$

more precisely, for $f \in \bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$ there exists a generalized derivative $D^\nu f$ and the following inequalities are valid

$$\|D^\nu f\|_{q,G} \le C_1 \sum_{i=0}^{n} |Q_T^i| \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta;G} \tag{27}$$

$$\|D^\nu f\|_{p,\psi,\beta^1;G} \le C_2 \|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G)}, \ (p^i \le p < \infty), \tag{28}$$

In particular, if

$$Q_{T,0}^i = \int_0^T \prod_{j=1}^{n} (\varphi_j(t))^{l_j^i - \nu_j - (1-\beta_i p)\frac{1}{p}} \prod_{j \in e_{l^i}} \frac{\varphi_j'(t)}{\varphi_j(t)} dt < \infty.$$

$i = 1,2,\dots,n$, then $D^\nu f(x)$ is continuous on G, i.e.

$$\sup_{x \in G} |D^\nu f(x)| \le C_1 \sum_{j=1}^{n} |Q_{T,0}^i| \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta;G} \tag{29}$$

$0 < T \le \min\{1, T_0\}$, T_0 is a fixed number; C_1 and C_2, are the constants independent of f, C_1 is independent also on T.

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^\nu f$ on G. Indeed, from the condition $Q_T^i < \infty$ for all $(i = 1,2,\dots,n)$, it follows that for $f \in \bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G) \hookrightarrow \bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i}(G)$, there exists $D^\nu f \in L_{p^i}(G)$ and for it integral representation (5) and (6) with the same kernels is valid.

Applying the Minkowskii inequality, from identities (5) and (6) we get

$$\|D^\nu f\|_{p,G} \le \left\|f^{(\nu)}_{\varphi(T)}\right\|_{p,G} + \sum_{i=1}^{n} \left\|Z^i_T\right\|_{q,G}. \tag{30}$$

By means of inequality (18) for $U = G$, $D^{l^i} f = \Phi$ $r = p$, $p^i = p^0$ we get

$$\left\|f^{(\nu)}_{\varphi(T)}\right\|_{p,G} \le C_1 \|f\|_{p^0,\varphi,\beta;G} \prod_{j=1}^{n} (\varphi_j(T))^{l^0_j - \nu_j - \left(1-\beta_j p^0\right)\frac{1}{p^0} - \frac{1}{p}} \times$$

$$\times \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j \frac{p^0}{p}} \le C_2 |Q^0_T| \left\|D^{l^0} f\right\|_{p^0,\varphi,\beta;G}, \tag{31}$$

and by means of inequality (9) for $\eta = T$, $U = G$, $L = M^{(\nu)}$,$D^{l^i} f = \Phi$, $\xi \to \infty$ we get

$$\left\|Z^i_T\right\|_{p,G} \le C_3 |Q^i_T| \left\|D^{l^i} f\right\|_{p^i,\varphi,\beta;G}. \tag{32}$$

Substituting (5) and (6) in (4), we get inequaltiy (27). By means of inequalities (22) for $U = G$, $D^{l^i} f = \Phi$, $p^i = p^0$ and (23) for $U = G$, $D^{l^i} f = \Phi$, $\eta = T$, we get inequality (28).

Now let conditions $Q^i_{T,0} < \infty$ $(i = 1, 2, \ldots, n)$, be satisfied, then based around identities (5) and (6), from inequality (6) for $p = \infty$ we get

$$\left\|D^\nu f - f^{(\nu)}_{\varphi(T)}\right\|_{\infty,G} \le C \sum_{i=1}^{n} |Q^i_{T,0}| \left\|D^{l^i} f\right\|_{p^i,\varphi,\beta;G}.$$

As $T \to 0$, the left side of this inequality tends to zero, since $f_{\varphi(T)}(x)$ is continuous on G and the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^\nu f$ is continuous on G.

This complete the proof of Theorem 2.1.

Let γ be an n-dimensional vector.

Theorem 2.2 *Let all the conditions of theorem 1 be fulfilled. Then for* $Q^i_T < \infty$ $(i = 1, 2, \ldots, n)$ *the generalized derivatives* $D^\nu f$ *satisfies on* G *the generalized Holder condition, i.e. the following inequality is valid:*

$$\|\Delta(\gamma, G) D^\nu f\|_{p,G} \le C \|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G)} |H(|\gamma, \varphi; T|)|, \tag{33}$$

where C *is a constant independent of* f, $|\gamma|$ *and* T.

In particular, $Q^i_{T,0} < \infty$ $(i = 1, 2, \ldots, n)$ *, then*

$$\sup_{x\in G} |\Delta(\gamma, G) D^\nu f(x)| \le C \|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G)} |H_0(|\gamma, \varphi; T|)|. \tag{34}$$

where

$$|H(|\gamma, \varphi; T|)| = \max_i \{|\gamma|, Q^i_{|\gamma|}, Q^i_{|\gamma|,T}\}, (|H_0(|\gamma, \varphi; T|)| = \max_i \{|\gamma|, Q^i_{|\gamma|,0}, Q^i_{|\gamma|,T,0}\}).$$

Proof. According to Lemma 8.6 from [2] there exists a domain

$$G_\omega \subset G\,(\omega = \zeta r\,(x)\,,\, \zeta > 0,\; r\,(x) = \rho\,(x, \partial G)\,,\; x \in G)$$

and assume that $|\gamma| < \omega$, then for any $x \in G_\omega$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identities (5) and (6) with the same kernels are valid. After same transformations, from (5) and (6) we get

$$|\Delta\,(\gamma, G)\, D^\nu f\,(x)| \le \prod_{j=1}^{n} (\varphi_j\,(T))^{l_j^0-\nu_j-1} \times$$

$$\int_{R^n} D^{l^0} f\,(x+y) \left|\Omega^{(\nu)}\left(\frac{y-\gamma}{\varphi\,(t)}, \frac{\rho\,(\varphi\,(t)\,, x)}{\varphi\,(t)}\right) - \Omega^{(\nu)}\left(\frac{y}{\varphi\,(t)}, \frac{\rho\,(\varphi\,(t)\,, x)}{\varphi\,(t)}\right)\right| dy +$$

$$\sum_{i=1}^{n} \left\{ \int_0^{|\gamma|} \int_{R^n} \left(\left|D^{l^i} f(x+y+\gamma)\right| + \left|D^{l^i} f(x+y)\right| \right) \times \right.$$

$$\left| M_i^{(\nu)}\left(\frac{y}{\varphi\,(t)}, \frac{\rho\,(\varphi\,(t)\,, x)}{\varphi\,(t)}, \rho'\,(\varphi\,(t)\,, x)\right)\right| \prod_{j=1}^{n} (\varphi_j\,(t))^{l_j^i-\nu_j-1} \prod_{j\in e^{l^i}} \frac{\varphi_j'\,(t)}{\varphi_j\,(t)} dt dy +$$

$$\int_{|\gamma|}^{T} \int_{R^n} \left|D^{l^i} f(x+y)\right| \left| M_i^{(\nu)}\left(\frac{y-\gamma}{\varphi\,(t)}, \frac{\rho\,(\varphi\,(t)\,, x)}{\varphi\,(t)}, \rho'\,(\varphi\,(t)\,, x)\right) - \right.$$

$$\left. \left. M_i^{(\nu)}\left(\frac{y}{\varphi\,(t)}, \frac{\rho\,(\varphi\,(t)\,, x)}{\varphi\,(t)}, \rho'\,(\varphi\,(t)\,, x)\right)\right| \prod_{j=1}^{n} (\varphi_j\,(t))^{l_j^i-\nu_j-1} \prod_{j\in e^{l^i}} \frac{\varphi_j'\,(t)}{\varphi_j\,(t)} dt dy \right\} =$$

$$= A(x,\gamma) + \sum_{i=1}^{n} (B_i(x,\gamma) + F_i(x,\gamma)), \tag{35}$$

where $0 < T \le \min\{1, T_0\}$, we also assume that $|\gamma| < T$. Consequently, $|\gamma| < \min(\omega, T)$. If $x \in G \setminus G_\omega$ then by definition

$$\Delta\,(\gamma, G)\, D^\nu f\,(x) = 0.$$

Based around (35) we have

$$\|\Delta\,(\gamma, G)\, D^\nu f\|_{p,G} \le \|A(x,\gamma)\|_{p,G_\omega} +$$

$$+ \sum_{i=1}^{n} \left(\|B_i(x,\gamma)\|_{p,G_\omega} + \|F_i(x,\gamma)\|_{p,G_\omega} \right), \tag{36}$$

$$A(x,\gamma) \le$$

$$\leq \prod_{j=1}^{n} (\varphi_j(t))^{l_j^0-\nu_j-2} \int_0^{|\gamma|} \int_{R^n} |f(x+\zeta e_\gamma + y)| \left| D_j \Omega^{(\nu)} \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)} \right) \right| dy.$$

similarly we get

$$B_i(x,\gamma) \leq \int_0^{|\gamma|} \prod_{j=1}^{n} (\varphi_j(t))^{l_j^i-\nu_j-2} \prod_{j\in e} \frac{\varphi_j'(t)}{\varphi_j(t)} dt \times$$

$$\int_{R^n} \left| D^{l^i} f(x+\zeta e_\gamma + y) \right| \left| D_j \Omega^{(\nu)} \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)} \right) \right| dy.$$

Taking into account $\xi e_\gamma + G_\omega \subset G$, and applying the generalized Minkowski inequality, from inequality (18) for $U = G$, $D^{l_0} f = \Phi$, $r = p$, $p^i = p^0$, $\xi \to \infty$ we have

$$\|A(\cdot,\gamma)\|_{p,G_\omega} \leq C_1 |\gamma| \, \|f\|_{p^0,\varphi,\beta;G}. \tag{37}$$

By means of inequality (9), for $U = G$, $D^{l_e} f = \Phi$, $\eta = |\gamma|$, $\xi \to \infty$ we get

$$\|B(\cdot,\gamma)\|_{p,G_\omega} \leq C_2 |Q^i_{|\gamma|}| \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta;G}. \tag{38}$$

and by means of inequality (36) for $U = G$, $D^{l^e} f = \Phi$, $\eta = |\gamma|$,$\xi \to \infty$ we get

$$\|F(\cdot,\gamma)\|_{p,G_\omega} \leq C_3 |Q^i_{|\gamma|,T}| \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta;G}. \tag{39}$$

From inequalities (36) and (37) - (39) we get the required inequality.

Now suppose that $|\gamma| \geq \min(\gamma, T)$. Then

$$\|\Delta(\gamma, G) D^\nu f\|_{p,G} \leq 2 \|D^\nu f\|_{p,G} \leq C(\varepsilon, T) \|D^\nu f\|_{p,G} H(|\gamma, \varphi; T|).$$

Estimating for $\|D^\nu f\|_{p,G}$ by means of inequality (27), in this case we get estimation (33).

Theorem 2.2 is proved.

References

[1] A. Akbulut, A. Eroglu and A.M.Najafov, *Some Embedding Theorems on the Nikolskii-Morrey Type Spaces*, Adv. in Anal., **1**, (2016), no. 1, 18–26.

[2] O.V. Besov, V.P. Il'yin and S. M.Nikolskii, *Integral representations of functions and embeddings theorems*, M. Nauka, (1996).

[3] V.S.Guliyev, *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J., **3** (2012), no. 3, 33–61.

[4] V.S. Guliyev and M.N. Omarova, *Multilinear singular and fractional integral operators on generalized weighted Morrey spaces*, Azerb. J. Math., **5**, (2015), no. 1 104–132.

[5] D.I. Hakim, Y. Sawano and T. Shimomura, *Boundedness of Generalized Fractional Integral Operators From the Morrey Space $L_{1,\phi}(X;\mu)$ to the Campanato Space $L_{1,\psi}(X;\mu)$ Over Non-doubling Measure Spaces*, Azerb. J. Math. **6**,(2016),no. 2, 117–127.

[6] V.P. Il'yin, *On some properties of the functions of spaces $W^{l}_{p,a,\chi}(G)$*, Zap. Nauch.Sem. LOMI AN USSR, **2**, 1971, 33–40.

[7] A.D.Jabrailov, Imbedding theorems for a space of functions with mixed derivatives satisfying the Hölder's multiple inteqral condition. Trudy MIAN SSSR, 117(1972), 113-138 (in Russian)

[8] L. Sh. Kadimova and A. M. Najafov, *Theorems on imbedding of functions from the Sobolev-Morrey generalized space.* Proceedings of A. Razmadze Mathematical Institute. Georgia 2010, 97-109.

[9] V.Kokilashvili, A. Meskhi and H. Rafeiro, *Sublinear operators in generalized weighted Morrey spaces*, Dokl. Math. **94**, (2016), no.2, 558–560.

[10] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, (1938), 126–166.

[11] A.M.Najafov The embedding theorems of space $W^{l}_{p,\varphi,\beta}(G)$. Mathematica Aeterna, Vol. 3, 2013, no. 4,pp. 299 - 308.

[12] A.M. Najafov, *On Some Properties of Functions in the Sobolev-Morrey-Type Spaces $W^{l}_{p,a,\kappa,\tau}(G)$*, Sib. Math. J.,**46**, (2005), no.3, 501-513

[13] A.M.Najafov and A.T. Orujova, *On properties of the generalized Besov-Morrey spaces with domiant mixed derivatives*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **41**, (2015), no. 1, pp. 3–15.

[14] E. Nakai, *Generalized fractional integrals on generalized Morrey spaces*, Math. Nachr. **287**, (2014),no.2-3, 339–351.

[15] Yu. V. Netrusov, *On some imbedding theorems of Besov-Morrey type spaces* Zap. Nauch.Sem. LOMI AN USSR, **139** (1984), 139–147 (in Russian).

[16] I. Ross, *A Morrey-Nicolskii inequality*, Proc. Amer. Math. Soc. **78**,(1980), 97–102.

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