Mathematica Aeterna, Vol. 7, 2017, no. 3, 211 - 221

The matrix in semilinear spaces over commutative semirings

Wenxin Luo

School of Mathematics and Statistics
Northeast Normal University
Changchun, 130024, CHINA

Chunchan Weng

School of Mathematics and Statistics
Northeast Normal University
Changchun, 130024, CHINA

Abstract

In this paper, some properties of matrices over commutative semirings are researched deeply. We extend the theorem about invertible matrix and show a necessary condition that a matrix is invertible.And we discuss in n-dimensional L-semilinear space V_n every vector of V_n can be uniquely represented by a linear combination of any basis of V_n. On the other hand, we show the connection between two bases of V_n with the transition matrix and prove an inequality in case that the rank of the matrix is redefined over commutative semirings. We give the proof that a set of linearly independent vectors is still linearly independent under semilinear transformation. We prove that some theorems of the determinant of a matrix still exist for the permanent, but some of the theorems do not. We show the necessary and sufficient condition that the permanent of an invertible matrix is zero.

Mathematics Subject Classification: Algebra

Keywords: commutative semirings, semilinear space, matrix, basis, semilinear transformation

1 Introduction

The study of semilinear structures over commutative semirings has a long history. In the theory of matrices over semirings, an invertible matrix is an important type of matrices. In 1984, Reutenauer and Straubing researched the invertible matrices over commutative semirings [9]. Moreover, in 2011, Shu and

Wang showed some necessary and sufficient conditions that each basis has the same number of elements over commutative zerosumfree semirings and proved that a set of vectors is a basis if and only if they are standard orthogonal [4]. In 2015, Zhang Houjun and Chu Maoquan researched the dimension of semilinear space over commutative semirings and got a series of results [7].

This paper is organized as follows. Some properties of matrices over commutative semirings are researched deeply. Firstly we extend the theorem about invertible matrix and show a necessary condition that a matrix is invertible. On the other hand, we show the connection between two bases of V_n with the transition matrix and prove an inequality in case that the rank of the matrix is redefined over commutative semirings. Then we prove that some theorems of the determinant of a matrix still exist for the permanent, but some of the theorems don't. We show the necessary and sufficient condition that the permanent of an invertible matrix is zero.

2 Preliminary Notes

Definition 2.1[1] *A semiring $L =< L, +, \cdot, 0, 1 >$ is an algebraic structure with the following properties:*

(1) $(L, +, 0)$ *is a commutative monoid,*

(2) $(L, \cdot, 1)$ *is a monoid,*

(3) $r \cdot (a + b) = r \cdot a + r \cdot b$ *and* $(a + b) \cdot r = a \cdot r + b \cdot r$ *hold for all* $a, b, r \in L$,

(4) $r \cdot 0 = 0 \cdot r = 0$ *hold for all* $r \in L$,

(4) $0 \neq 1$,

A semiring L is commutative if $r \cdot r' = r' \cdot r$ for all $r, r' \in L$.

Natural number and the set of nonnegative real number with the usual operations of addition and multiplication of real numbers are commutative semirings.

Definition 2.2[1] *Let $L =< L, +, \cdot, 0, 1 >$ be a semiring and let $A =< A, +_A, 0_A >$ be a commutative monoid. If $* : L \times A \to A$ is an external multiplication such that*

(1) $(r \cdot r') * a = r \cdot (r' * a)$,

(2) $r * (a +_A a') = r * a +_A r * a'$,

(3) $(r + r') * a = r * a +_A r' * a$,

(4) $1 * a = a$,

(5) $r * 0_A = 0 * a = 0$,

for all $r, r' \in L$ and $a, a' \in A$ then $< L, +, \cdot, 0, 1, *, A, +_A, 0_A >$ is called a left L-semimodule. The definition of a right L-semimodule is analogous, where the external multiplication is defined as a function $L \times A \to A$.

The following definition is a general version of a semilinear space in [10]:

Let $L =< L, +, \cdot, 0, 1 >$ be a semiring. Then a semimodule over L is called an L-semilinear space.

Note that a semimodule stands for a left L-semimodule and right L-semimodule as in [10]. Elements of an L-semilinear space will be called vectors and elements of a semiring scalars. The former will be denoted by hold letters to distinguish them from scalars.

Without loss of generality, in what follows, we consider left L-semimodules for convenience of notation. Then we can construct an L-semilinear space as follows.

Natural number $Z^0 =< Z^0, +, \cdot, 0, 1 >$ is a semiring. For $\forall n \geq 1$, let

$V_n(Z^0) = \{(a_1, a_2, \cdots, a_n)^T : a_i \in Z^0, i = 1, \cdots, n.\}$,

$\mathbf{x}= (x_1, x_2, \cdots, x_n)^T$,$\mathbf{y}= (y_1, y_2, \cdots, y_n)^T \in V_n(Z^0)$, $r \in Z^0$,

definite $\mathbf{x}$+$\mathbf{y}$= $(x_1 + y_1, \cdots, x_n + y_n)^T$,$r * \mathbf{x}= (r \cdot x_1, r \cdot x_2, \cdots, r \cdot x_n)^T$. Then $V_n(Z^0)$ is a semilinear space and $\mathbf{0}_{n\times 1} = (0, 0, \cdots, 0)^T$.

Definition 2.3[5] *Let V_n be an L-semilinear space. The expression where $\lambda_1, \lambda_2, \cdots, \lambda_n \in L$ are scalars is called a linear combination of vectors $\boldsymbol{\alpha}_1$,$\boldsymbol{\alpha}_2$,$\cdots$,$\boldsymbol{\alpha}_n$.*

If the vector $\mathbf{x}$ can be expressed as a linear combination by the vector set $\boldsymbol{\alpha}_1$,$\boldsymbol{\alpha}_2$,$\cdots$,$\boldsymbol{\alpha}_n$, then we say vector $\mathbf{x}$ can be expressed by the vector set $\boldsymbol{\alpha}_1$,$\boldsymbol{\alpha}_2$,$\cdots$,$\boldsymbol{\alpha}_n$ in a linear form.

Definition 2.4[5] *In an L-semilinear space, vectors $\boldsymbol{\alpha}_1$,$\boldsymbol{\alpha}_2$,$\cdots$,$\boldsymbol{\alpha}_n (n \geq 2)$ are linearly independent if none of them can be represented by a linear combination of the others. Otherwise, we say that vectors $\boldsymbol{\alpha}_1$,$\boldsymbol{\alpha}_2$,$\cdots$,$\boldsymbol{\alpha}_n$ are linearly dependent.*

In semilinear space V_n, let $\boldsymbol{\alpha}_1$,$\boldsymbol{\alpha}_2$,$\cdots$,$\boldsymbol{\alpha}_n$ and $\boldsymbol{\beta}_1$,$\boldsymbol{\beta}_2$,$\cdots$,$\boldsymbol{\beta}_m$ be the two sets of vectors.If every $\boldsymbol{\alpha}_i \in \underline{n}$ can be represented by a linear combination of $\boldsymbol{\beta}_1$,$\boldsymbol{\beta}_2$,$\cdots$,$\boldsymbol{\beta}_m$ and every $\boldsymbol{\beta}_j \in \underline{m}$ can be represented by a linear combination of $\boldsymbol{\alpha}_1$,$\boldsymbol{\alpha}_2$,$\cdots$,$\boldsymbol{\alpha}_n$, then the two sets of vectors are said to be equivalent.

If a part of vector $\boldsymbol{\alpha}_1$,$\boldsymbol{\alpha}_2$,$\cdots$,$\boldsymbol{\alpha}_n$ is linearly independent, then the vector set $\boldsymbol{\alpha}_1$,$\boldsymbol{\alpha}_2$,$\cdots$,$\boldsymbol{\alpha}_n$ is linearly independent. If the vector set $\boldsymbol{\alpha}_1$,$\boldsymbol{\alpha}_2$,$\cdots$,$\boldsymbol{\alpha}_n$ is linearly independent, then any part of it is linearly independent.

Definition 2.5[5] *A nonempty subset G of an L-semilinear space is called a set of generators if every element of the L-semilinear space is a linear combination of elements in G. Let S be a set of generators of L-semilinear space V_n . Then we put $V_n =< S >$.*

Definition 2.6[1] *Suppose V_n is a semilinear space, a set of linearly independent generators is called the basis of V_n.*

Obviously, V_n is a semilinear space with a basis $\mathbf{e}_1$,$\mathbf{e}_2$,$\cdots$,$\mathbf{e}_n$,where

$\mathbf{e}_1 = (1, 0, \cdots, 0)^T$,$\mathbf{e}_2 = (0, 1, \cdots, 0)^T$,$\cdots$,$\mathbf{e}_n = (0, 0, \cdots, 1)^T$.

Note that in [6], we call $\mathbf{e}_1$,$\mathbf{e}_2$,$\cdots\mathbf{e}_n$ is the standard basis of semilinear space.

Different from the linear space, in general, the cardinality of basis is not unique.

Let $M_{m\times n}(L)$ be the set of all $m\times n$ matrices over a semiring $< L, +, \cdot, 0, 1 >$. In particular, let for $M_n(L) = M_{n\times n}(L)$.Given $A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n} \in M_{m\times n}(L)$ and $C = (c_{ij})_{n\times l} \in M_{n\times l}(L)$, we define that $A + B = (a_{ij} + b_{ij})_{m\times n}$, $AC = (\sum\limits_{k=1}^{n} a_{ik} \cdot c_{kj})_{m\times l}$,$\lambda A = (\lambda a_{ij})_{m\times n}$, $\forall\lambda \in L$.

Definition 2.7[3] *If each basis of an L-semilinear space V_n has the same number of elements, then we call the number of the vectors in each basis a dimension of V_n , in symbols $dim(V_n)$.*

Definition 2.8[7] *A matrix $A \in M_n(L)$ is called right(left) invertible if there is a matrix $B \in M_n(L)$ such that $AB = I_n(BA = I_n)$. If the matrix A s not only left invertible but also right invertible, then we call A is invertible.*

Definition 2.9[8] *If $A \in M_n(L)$, $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$,*

then define the positive and negative determinants as $|A|^+ = \sum\limits_{\sigma=even} a_{1,j_1}a_{2,j_2}\cdots a_{n,j_n}$ and $|A|^- = \sum\limits_{\sigma=odd} a_{1,j_1}a_{2,j_2}\cdots a_{n,j_n}$. We note that the permanent of A is given by $per(A) = |A|^+ + |A|^-$ and the determinant of A is given by $det(A) = |A|^+ - |A|^-$.

Definition 2.10[4] *Let V_n be a semilinear space on semirings, $\mathbf{x}= (x_1, x_2, \cdots, x_n)^T$,$\mathbf{y}= (y_1, y_2, \cdots, y_n)^T$, defined $(\mathbf{x}, \mathbf{y}) =x_1y_1 + x_2y_2 + \cdots + x_ny_n$ is the inner product of $\mathbf{x}$ and $\mathbf{y}$.*

Definition 2.11[7] *Let $\mathbf{x}$ and $\mathbf{y}$ be the two vectors of the semilinear space if $(\mathbf{x}, \mathbf{y}) =0$, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal to each other.*

Definition 2.12[7] *Let $\{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n\}$ be the basis of the semilinear space V_n, if $\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n$ are all orthogonal to each other, then $\{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n\}$ is the orthogonal basis of V_n.*

Let $\{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n\}$ and $\{\mathbf{y}_1,\mathbf{y}_2,\cdots,\mathbf{y}_m\}$ be the two bases of V_n, and each element of $\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n$ is linear combination of $\mathbf{y}_1,\mathbf{y}_2,\cdots,\mathbf{y}_m$,namely

$$\begin{cases} \mathbf{x}_1 = a_{11}\mathbf{y}_1 + a_{12}\mathbf{y}_2 + \cdots + a_{1m}\mathbf{y}_m \\ \mathbf{x}_2 = a_{21}\mathbf{y}_1 + a_{22}\mathbf{y}_2 + \cdots + a_{2m}\mathbf{y}_m \\ \cdots \\ \mathbf{x}_n = a_{n1}\mathbf{y}_1 + a_{n2}\mathbf{y}_2 + \cdots + a_{nm}\mathbf{y}_m \end{cases}$$

or $(\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n)= (\mathbf{y}_1,\mathbf{y}_2,\cdots,\mathbf{y}_m)A$, $A = (a_{ij})_{m\times n} \in M_{m\times n}(L)$, then matrix A is called a transition matrix from $\{\mathbf{y}_1,\mathbf{y}_2,\cdots,\mathbf{y}_m\}$ to $\{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n\}$, The rank of matrix $A \in M_{m\times n}(L)$ is k if there exist $B \in M_{m\times k}(L)$ and $C \in M_{k\times n}(L)$

such that $A = BC$ and k is the least positive integer . Then we have $f(A) = k$ in symbols.

Note that if $A \in M_n(L)$, then $f(A) \leq n$.

Definition 2.13[7] *A transformation φ on semilinear space V_n is called semilinear transformation, if for any vector $\mathbf{x}$,$\mathbf{y}$$\in V_n$, $k \in L$, we have $\varphi(\mathbf{x}+\mathbf{y})$= $\varphi(\mathbf{x})$+$\varphi(\mathbf{y})$, $\varphi(k\mathbf{x})$= $k\varphi(\mathbf{x})$.*

Let $\mathbf{x}_1$,$\mathbf{x}_2$,$\cdots$,$\mathbf{x}_n$ be a set of vectors of semilinear space V_n, φ is a semilinear transformation of V_n, then if

$$\begin{cases} \varphi(\mathbf{x}_1) = a_{11}\mathbf{x}_1 + a_{12}\mathbf{x}_2 + \cdots + a_{1n}\mathbf{x}_n \\ \varphi(\mathbf{x}_2) = a_{21}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + \cdots + a_{2n}\mathbf{x}_n \\ \cdots \\ \varphi(\mathbf{x}_n) = a_{n1}\mathbf{x}_1 + a_{n2}\mathbf{x}_2 + \cdots + a_{nn}\mathbf{x}_n \end{cases}$$

or $\varphi(\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n)$= $(\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_m)A$,$A = (a_{ij})_{m\times n} \in M_{m\times n}(L)$,then we call A is the matrix of transformation φ on $\mathbf{x}_1$,$\mathbf{x}_2$,$\cdots$,$\mathbf{x}_n$.

Definition 2.14[7] *Let L be a commutative semiring and $A, B \in M_n(L)$, A is similar to B if there is an invertible matrix $X \in M_n(L)$ such that $A = X^{-1}BX$.*

3 Main Results

These are the main results of the paper.

Lemma 3.1[9] *Let L be a commutative semiring and $A, B \in M_n(L)$, if $AB = I_n$, then $BA = I_n$.*

Lemma 3.2[7] *Let V_n be a semilinear space, then $dim(V_n) = n$ if and only if every vector of V_n can be uniquely represented by a linear combination of standard orthogonal basis.*

Theorem 3.1 *Let L be a commutative semiring and $A_1, A_2, \cdots, A_n \in M_n(L)$, if $A_1A_2\cdots A_n = I_n$, then $A_1, A_2, \cdots, A_n$ are invertible matrices.*

Proof. Since $A_1A_2\cdots A_n = I_n$, A_1 is invertible.

By Lemma 1, we have $A_1A_2\cdots A_n = A_2\cdots A_nA_1 = I_n$,then A_2 is invertible.

Similarly, $A_3, A_4, \cdots, A_n$ are invertible matrices. □

Note that $A_1A_2\cdots A_n = A_2\cdots A_nA_1 = \cdots = A_nA_1\cdots A_{n-1} = I_n$.

Theorem 3.2 *Let V_n be a semilinear space and $dim(V_n) = n$. If $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_n\}$ is the basis of semilinear space V_n, then every vector of V_n can be uniquely represented by a linear combination of $\boldsymbol{\alpha}_1$,$\boldsymbol{\alpha}_2$,$\cdots$,$\boldsymbol{\alpha}_n$*

Proof. Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be standard orthogonal basis of semilinear space V_n.

Since $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$ is the basis of V_n, there is a matrix $A = (a_{ij}) \in M_n(L)$ such that $(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n) = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n)A$.

From Lemma 2, we know that A is unique and invertible.

For all $\boldsymbol{\alpha} \in V_n$, there exists elements $k_1, k_2, \cdots, k_n \in L$ such that

$\boldsymbol{\alpha} = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n)(k_1, k_2, \cdots, k_n)^T$.

Then we have $\boldsymbol{\alpha} = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n)(k_1, k_2, \cdots, k_n)^T = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n)A(k_1, k_2, \cdots, k_n)^T$.

Therefore, every vector of V_n can be uniquely represented by a linear combination of $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$. □

Corollary *Let V_n be a semilinear space and $dim(V_n) = n$. If $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$ and $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n\}$ are the bases of V_n, then $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$ and $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n\}$ are equivalent.*

Theorem 3.3 *Let V_n be a semilinear space and $dim(V_n) = n$. If $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$ and $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n\}$ are the bases of V_n and $A = (a_{ij}) \in M_n(L)$ such that $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n) = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n)A$, then A is invertible.*

Proof. Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be standard orthogonal basis of semilinear space V_n.

Since $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$ and $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n\}$ are the bases of V_n, there are matrices $B, C \in M_n(L)$ such that

$(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n) = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n)B$ and $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n) = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n)C$.

Then we have

$(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n) = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n)B = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n)AB = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n)CAB$.

From Theorem 1, we know that A is invertible. □

Theorem 3.4 *Let L be a commutative semiring and $A \in M_n(L)$,if A is invertible, then $f(A) = n$.*

Proof. Since A is invertible, there exists matrix $B \in M_n(L)$ such that $AB = I_n$.

Assume that $k \leq n$, then there exists matrix $C \in M_{n\times k}(L)$ and $D \in M_{k\times n}(L)$ such that $A = CD$.

Therefore, $CDB = I_n$.

If $k < n$, we add $n-k$ columns $\mathbf{0}$ to C and add $n-k$ rows $\mathbf{0}$ to DB. Then we have $\begin{pmatrix} C & \mathbf{0} \end{pmatrix} \begin{pmatrix} DB \\ \mathbf{0} \end{pmatrix} = I_n$.

From Definition 8 and Lemma 2, both $\begin{pmatrix} C & \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} DB \\ \mathbf{0} \end{pmatrix}$ are invertible matrices and $\begin{pmatrix} C & \mathbf{0} \end{pmatrix} \begin{pmatrix} DB \\ \mathbf{0} \end{pmatrix} = I_n$.

On the other hand, we know that $\begin{pmatrix} C & \mathbf{0} \end{pmatrix} \begin{pmatrix} DB \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} DBC & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \neq I_n$.

Therefore, $k = n$.

Consequently,$f(A) = n$. □

Lemma 3.3[9] *Let $A \in M_n(L)$.If A is invertible, then the column vectors of A are linearly independent.*

Theorem 3.5 *Let V_n be a semilinear space, $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m\}$ is the basis of V_n. If there is $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r\}$ and a matrix A, such that $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r)A = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m)$ and $f(A) = n$, then $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r\}$ is a basis of V_n.*

Proof. $f(A) = n$ and there is $B \in M_{r\times n}(L)$ and $C \in M_{n\times m}(L)$, s.t. $A = BC$.

$$(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r)A = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r)BC = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m)$$

Let $\boldsymbol{\gamma}_l = \sum\limits_{j=1}^{r} b_{jl}\boldsymbol{\beta}_j$, $l \in 1, 2, \cdots, n$, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m\}$ is the standard basis of V_n. Since $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m) = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_n)C$, $\{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_n\}$ is a set of generators of V_n.

Therefore, $\boldsymbol{e}_i$ can be represented by a linear combination of $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_n$, $i = 1, 2, \cdots, n$. $I_n = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_n) \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{pmatrix}$, then $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_n)$ is right invertible.

From Lemma 1 we know that $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_n)$ is invertible. According to Lemma 3, $\{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_n\}$ is a basis of V_n. $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_n) = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r)B$, then $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r\}$ is a set of generators of V_n.

In the same way, $I_n = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r) \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{r1} & \cdots & \lambda_{rn} \end{pmatrix}$, then $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r)$ is right invertible. From Lemma 1 we know that $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r)$ is invertible. According to Lemma 3, $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r\}$ is a basis of V_n. □

Corollary *Let V_n be a semilinear space, $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m\}$ is the basis of V_n. If there are $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r\}$ and a matrix A, such that $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r)A = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m)$ and A is invertible, then $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r\}$ is a basis of V_n.*

Theorem 3.6 *Suppose $A = (a_{ij}) \in M_{m\times n}(L)$ and $B = (b_{ij}) \in M_{n\times s}(L)$, then $f(AB) \leq min\{f(A), f(B)\}$.*

Proof. Let $f(AB) = k$, $f(A) = a$ and $f(B) = b$. Then $A = A_1A_2$ with $A_1 = (a^1_{ij}) \in M_{m\times a}(L)$, $A_2 = (a^2_{ij}) \in M_{a\times n}(L)$. $B = B_1B_2$ with $B_1 = (b^1_{ij}) \in M_{n\times b}(L)$, $B_2 = (b^2_{ij}) \in M_{b\times s}(L)$.

Then $AB = A_1A_2B_1B_2 = CB_2 = A_1D$, with $C = (c_{ij}) \in M_{m\times b}(L)$, $B_2 = (b_{ij}^2) \in M_{b\times s}(L)$, $A_1 = (a_{ij}^1) \in M_{m\times a}(L)$ and $D = (d_{ij}) \in M_{a\times s}(L)$. From Definition 13, $k \leq b$ and $k \leq a$. Therefore, $f(AB) \leq min\{f(A), f(B)\}$. □

Theorem 3.7 *φ is a semilinear transformation on semilinear space V_n and $\varphi^{-1}(\boldsymbol{0}) = \boldsymbol{0}$, then $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_l\}$ is linearly independent if and only if $\{\varphi(\boldsymbol{\alpha}_1), \varphi(\boldsymbol{\alpha}_2), \cdots, \varphi(\boldsymbol{\alpha}_l)\}$ is linearly independent.*

Proof. ($\Rightarrow$) Let $k_1\varphi(\boldsymbol{\alpha}_1) + k_2\varphi(\boldsymbol{\alpha}_2) + \cdots + k_l\varphi(\boldsymbol{\alpha}_l) = \mathbf{0}$. Then from Definition 13, we have $\varphi(k_1\boldsymbol{\alpha}_1 + k_2\boldsymbol{\alpha}_2 + \cdots + k_l\boldsymbol{\alpha}_l) = 0$.

Since $\varphi^{-1}(\mathbf{0}) = \mathbf{0}$, $k_1\boldsymbol{\alpha}_1 + k_2\boldsymbol{\alpha}_2 + \cdots + k_l\boldsymbol{\alpha}_l = \mathbf{0}$.

Since $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_l\}$ is linearly independent, $k_1 = k_2 = \cdots = k_l = 0$. Then we have $\{\varphi(\boldsymbol{\alpha}_1), \varphi(\boldsymbol{\alpha}_2), \cdots, \varphi(\boldsymbol{\alpha}_l)\}$ is linearly independent.

($\Leftarrow$) Let $k_1\boldsymbol{\alpha}_1 + k_2\boldsymbol{\alpha}_2 + \cdots + k_l\boldsymbol{\alpha}_l = \mathbf{0}$, then $\varphi(k_1\boldsymbol{\alpha}_1 + k_2\boldsymbol{\alpha}_2 + \cdots + k_l\boldsymbol{\alpha}_l) = k_1\varphi(\boldsymbol{\alpha}_1) + k_2\varphi(\boldsymbol{\alpha}_2) + \cdots + k_l\varphi(\boldsymbol{\alpha}_l) = \mathbf{0}$.

Since $\{\varphi(\boldsymbol{\alpha}_1), \varphi(\boldsymbol{\alpha}_2), \cdots, \varphi(\boldsymbol{\alpha}_l)\}$ is linearly independent, we have $k_1 = k_2 = \cdots = k_l = 0$. Therefore, $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_l\}$ is linearly independent. □

Lemma 3.4[8] *If $A \in M_n(L)$ and A has a zero row or column, then $|A|^+ = |A|^- = 0$.*

Lemma 3.5[8] *If $A \in M_n(L)$, $|A|^+ = |A^T|^+$ and $|A|^- = |A^T|^-$.*

Lemma 3.6[8] *If $A \in M_n(L)$ and suppose that B is obtained from A by interchanging two rows (columns), then $|A|^+ = |B|^-$ and $A|^- = |B|^+$.*

Lemma 3.7[8] *If $A \in M_n(L)$ and A has two equal rows (columns), then $|A|^+ = |A|^-$.*

Lemma 3.8[8] *Suppose $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n)$,*

(i) If $B = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{r-1}, \gamma\boldsymbol{a}_r, \boldsymbol{a}_{r+1}, \cdots, \boldsymbol{a}_n)$, then $|B|^\pm = \gamma|A|^\pm$.

(ii) If $C = \begin{pmatrix} \boldsymbol{a}_1^T \\ \vdots \\ \gamma\boldsymbol{a}_r^T \\ \vdots \\ \boldsymbol{a}_n^T \end{pmatrix}$, *then $|C|^\pm = \gamma|A|^\pm$.*

Lemma 3.9[8] *If $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n)$ and $\boldsymbol{a}_k = \boldsymbol{b}_k + \boldsymbol{c}_k$, $k = 1, \cdots, n$, then $|A|^\pm = |\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{b}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n|^\pm + |\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{c}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n|^\pm$.*

Lemma 3.10[8] *If* $M = \begin{pmatrix} A & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{pmatrix}$, *then*

(i)$|M|^+ = |A|^+|D|^+ + |A|^-|D|^-$,

(ii)$|M|^- = |A|^+|D|^- + |A|^-|D|^+$.

Lemma 3.11[8] *If $A, B \in M_n(L)$, then $|AB|^+ + |A|^+|B|^- + |A|^-|B|^+ =$*

$|AB|^- + |A|^+|B|^+ + |A|^-|B|^-$.

Theorem 3.8 *If A is invertible, then $per(A) = 0$ if and only if $|A|^+|A^{-1}| = \frac{1}{2}$.*

Proof. $(\Rightarrow)$ From Lemma 11, let B be A^{-1}, then we have

$$1 + |A|^+|A^{-1}|^- - |A|^+|A^{-1}|^+ = |A|^-|A^{-1}|^- - |A|^-|A^{-1}|^+,$$

that is $1 + |A|^+(|A^{-1}|^- - |A^{-1}|^+) = |A|^-(|A^{-1}|^- - |A^{-1}|^+)$,
so $(|A|^- - |A|^+)(|A^{-1}|^- - |A^{-1}|^+) = 1$.
Since $per(A) = 0$ if and only if $|A|^+ = -|A|^-$, that is $|A|^+(|A^{-1}|^+ - |A^{-1}|^-) = \frac{1}{2}$. So $|A|^+|A^{-1}| = \frac{1}{2}$.
$(\Leftarrow)$ By contrary, if $|A|^+|A^{-1}| = \frac{1}{2}$, then $|A|^+ = -|A|^-$.
And we have $per(A) = 0$. □

Theorem 3.9 *If $A \in M_n(L)$, then $per(A) = per(A^T)$.*

Proof. $per(A) = |A|^+ + |A|^-$ and $per(A^T) = |A^T|^+ + |A^T|^-$.
From Lemma 5 we know $|A|^+ + |A|^- = |A^T|^+ + |A^T|^-$, so $per(A) = per(A^T)$. □

Theorem 3.10 *If $A, B \in M_n(L)$ and B is obtained from A by interchanging two rows (columns), then $per(A) = per(B)$.*

Proof. Since $per(A) = |A|^+ + |A|^-$ and $per(B) = |B|^+ + |B|^-$,
we know $|B|^+ = |A|^-$ and $|B|^- = |A|^+$ from Lemma 6. So $per(A) = per(B)$. □

Theorem 3.11 *Suppose $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n)$,*

(i)If $B = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{r-1}, \gamma\boldsymbol{a}_r, \boldsymbol{a}_{r+1}, \cdots, \boldsymbol{a}_n)$, then $per(B) = \gamma per(A)$.

(ii) If $C = \begin{pmatrix} \boldsymbol{a}_1^T \\ \vdots \\ \gamma\boldsymbol{a}_r^T \\ \vdots \\ \boldsymbol{a}_n^T \end{pmatrix}$, *then $per(C) = \gamma per(A)$.*

Proof. (i)From Lemma 8, we know $|B|^+ = \gamma|A|^+$ and $|B|^- = \gamma|A|^-$,
then $per(B) = |B|^+ + |B|^- = \gamma per(A)$.
(ii)In a similar way, from Lemma 8 we have $per(C) = \gamma per(A)$. □

Theorem 3.12 *If $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n)$, and $\boldsymbol{a}_k = \boldsymbol{b}_k + \boldsymbol{c}_k$, $k = 1, \cdots, n$, then $per(A) = per(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{b}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n) + per(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{c}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n)$.*

Proof. From Lemma 4 we know that $|A|^+ = |\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{b}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n|^+ + |\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{c}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n|^+$,

$|A|^- = |\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{b}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n|^- + |\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{c}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n|^-$.

Then we have $per(A) = per(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{b}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n) + per(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{c}_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_n)$.

Note that (i)If $A = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_i, \cdots, \boldsymbol{a}_j, \cdots, \boldsymbol{a}_n)$, $B = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_i, \cdots, \boldsymbol{a}_j + k\boldsymbol{a}_i, \cdots, \boldsymbol{a}_n)$, $k \in R$, then different from $det(A) = det(B).per(A) \neq per(B)$.

(ii)A has two equal rows (columns), then $per(A) = |A|^+ + |A|^- = 2|A|^+$. □

Theorem 3.13 *If* $M = \begin{pmatrix} A & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{pmatrix}$, *then* $per(M) = per(A)per(D)$.

Proof. From Lemma 10 we know that

$per(M) = |M|^+ + |M|^- = |A|^+|D|^+ + |A|^-|D|^- + |A|^+|D|^- + |A|^-|D|^+$,

that is

$per(M) = |A|^+(|D|^+ + |D|^-) + |A|^-(|D|^+ + |D|^-) = per(D)(|A|^+ + |A|^-) = per(A)per(D)$,

then we have $per(M) = per(A)per(D)$. □

ACKNOWLEDGEMENTS.

References

[1] GOLAR J S., Semirings and their applications[M], Dordrecht:Kluwer Academic Publishers, 1999.

[2] DI-NOLAR A,LETTIERI A,PERFILIEVA I, Algebraic analysis of fuzzy systems[J], Fuzzy Sets and Systems, 2007, 158:1-22.

[3] SHU Qianyu,WANG Xueping, Bases in semilinear spaces over zerosumfree semirings[J], Linear Algebra Appl, 2011, 435:2681-2692.

[4] SHU Qianyu,WANG Xueping, Standard orthogonal vectors in semilinear space and their applications[J], Linear Algebra Appl, 2012, 437:2733-2754.

[5] DI-NOLAR A,LETTIERI A,PERFILIEVA I, Alagebraic analysis of fuzzy systems[J], Fuzzy Sets and Systems, 2007, 158:1-22.

[6] ZHAO Shan,WANG Xueping, Invertible matrices and semilinear spaces over commutative semirings[J], Information Sciences, 2010, 180:5115-5124.

[7] ZHANG Hou jun,CHU Mao quan, Dimensions of semilinear spaces over commutative semirings, Journal of Shandong University(Natural Science), 2015, 229:45-52.

[8] Phillip L.,Poplin,Robert E.,Hartwig, Determinantal identities over commutative semirings, Linear Algebra and its Applications, 2004, 387:99-132.

[9] REUTENAUER C.,STRAUBING H., Inversion of matrices over a commutative semiring[J], J Algebra, 1984, 88:350-360.

[10] A Di Nola,A.Lettieri,I. Perfilieva,V.Novak,Algebraic, analysis of fuzzy systems, Fuzzy Sets Syst, 2007, 158:1-22.

Received: July 05, 2017