Mathematica Aeterna, Vol. 7, 2017, no. 3, 223 - 231

Concircular Curvature Tensor on Generalized Sasakian Space Forms

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Abstract

The aim of this paper is to study concircular curvature tensor on generalized Sasakian space forms. Here we describe the ϕ-concircular flat, pseudo-concircular flat, quasi-concircular flat, ϕ-concircular semi-symmetric and concircular pseudo-symmetric conditions on generalized Sasakian space forms and obtained interesting results.

Mathematics Subject Classification: 53C25, 53D10, 53D15.

Keywords: generalized Sasakian space forms, ϕ-concircular flat, pseudo-concircular flat, quasi-concircular flat, concircular pseudo-symmetric, η-Einstein manifold.

1 Introduction

In 2004, Alegre, Blair and Carriazo [1] introduced the notion of generalized Sasakian space forms. A Sasakian manifold with constant ϕ-sectional curvature is a Sasakian space-form and it has a specific form of its curvature tensor. An almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be a generalized Sasakian-space-form if there exists differentiable functions f_1, f_2, f_3 such that the curvature tensor R of $M^{2n+1}$is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y \\ &\quad -g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ &\quad -\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1}$$

for any vector fields X, Y, Z on M^{2n+1}. In such cases the manifold will be written as $M^{2n+1}(f_1, f_2, f_3)$. This nature of manifold appears in the generalization of Sasakian space form by taking $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$, where c denotes constant ϕ-sectional curvature. In [8], Kim studied conformally flat and locally symmetric generalized Sasakian space forms. Also Avijit Sarkar and Ali Akbar [10] studied projective curvature tensor on generalized Sasakian space forms. In [4], De studied generalized Sasakian space forms satisfying certain conditions on the concircular curvature tensor. The notion of generalized Sasakian space forms have been weakened by many geometers such as [5, 11, 12, 13] and many others with different curvature tensors.

Further the concircular curvature tensor $\tilde{C}$ in an $(2n+1)$-dimensional Riemannian manifold (M^{2n+1}, g) is defined by [14, 15]:

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y], \tag{2}$$

for all $X, Y, Z \in M^{2n+1}$.

The present paper is organized as follows: In Section 2, we have provided some preliminary results that will be needed throughout the paper. In section 3, we describe ϕ-concircular flat generalized Sasakian space forms. In Sections 4 and 5 we study pseudo-concircular flat and quasi concircular flat generalized Sasakian space forms and it is shown that the manifold reduces to η-Einstein. In Section 6, we proved that a $(2n+1)$-dimensional generalized Sasakian space form is concircular pseudo-symmetric, then either $L_{\tilde{C}} = f_1 - f_3$ or the manifold is η-Einstein. Finally section 7 is devoted to the study of ϕ-concircular semi-symmetric generalized sasakian space forms.

2 Preliminaries

In an almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$, we have [2, 3]

$$\phi^2 X = -X + \eta(X)\xi, \phi\xi = 0, \tag{3}$$
$$\eta(\xi) = 1, \; g(X, \xi) = \eta(X), \; \eta(\phi X) = 0, \tag{4}$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{5}$$
$$g(\phi X, Y) = -g(X, \phi Y), \; g(\phi X, X) = 0, \tag{6}$$

where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form ,∇ is the levi-civita connection and g is a Riemannian metric.
In a $(2n + 1)$-dimensional generalized Sasakian space form, the following relations hold [1]:

$$R(X, Y)\xi = (f_1 - f_3)\{\eta(X)\xi - X\}, \tag{7}$$
$$\eta(R(X, Y)Z) = (f_1 - f_3)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \tag{8}$$
$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi, \tag{9}$$
$$Q\xi = 2n(f_1 - f_3)\xi, \tag{10}$$
$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n - 1)f_3)\,\eta(X)\eta(Y), \tag{11}$$
$$S(X, \xi) = 2n(f_1 - f_3)\,\eta(X), \tag{12}$$
$$r = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3, \tag{13}$$
$$S(\phi X, \phi Y) = S(X, W) - 2n(f_1 - f_3)\eta(X)\eta(W), \tag{14}$$

for any vector fields X, Y, Z on M^{2n+1} where R, Q, S and r are the Riemannian curvature tensor, Ricci operator, Ricci tensor and scalar curvature of $M^{2n+1}(f_1, f_2, f_3)$ respectively. Also in a generalized Sasakian space form, concircular curvature tensor satisfies the following:

$$\tilde{C}(X, Y)\xi = \left[f_1 - f_3 - \frac{r}{2n(2n + 1)}\right]\{\eta(Y)X - \eta(X)Y\}, \tag{15}$$
$$\tilde{C}(\xi, X)Y = \left[f_1 - f_3 - \frac{r}{2n(2n + 1)}\right]\{g(X, Y)\xi - \eta(Y)X\}, \tag{16}$$
$$\eta(\tilde{C}(\xi, X)\xi) = \left[f_1 - f_3 - \frac{r}{2n(2n + 1)}\right]\{\eta(X)\xi - X\}. \tag{17}$$

Definition 2.1 *A $(2n + 1)$-dimensional generalized sasakian space form is said to be η-Einstein if its Ricci tensor S is of the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

for any vector fields X and Y, where a and b are constants. If $b = 0$ then the manifold is Einstein and if $a = 0$ then the manifold is special type of η-Einstein.

3 ϕ-concircular flat generalized Sasakian space forms

Definition 3.1 *A $(2n+1)$-dimensional $(n \geq 1)$ generalized Sasakian space form is called ϕ-concircular flat if it satisfies $\phi^2(\tilde{C}(\phi X, \phi Y)\phi Z) = 0$, for every vector fields X, Y and Z.*

Let us consider a ϕ-concircular flat generalized Sasakian space form i.e.,

$$\phi^2(\tilde{C}(\phi X, \phi Y)\phi Z) = 0. \tag{18}$$

In view of (2) in (18), we get

$$\phi^2(R(\phi X, \phi Y)\phi Z) - \frac{r}{2n(2n+1)}\phi^2(g(\phi Y, \phi Z)\phi X - g(\phi X, \phi Z)\phi Y) = 0. \tag{19}$$

By using (1), (4) and (11) in (19), we obtain

$$\begin{aligned} &\phi^2[f_1(g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y) \\ &+f_2(g(X,\phi Z)\phi^2 Y - g(Y,\phi Z)\phi^2 X + 2g(X,\phi Y)\phi^2 Z)] \\ &= \frac{r}{2n(2n+1)}\phi^2[g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X \\ &-g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y]. \end{aligned} \tag{20}$$

By virtue of (3) and (4) in (20), gives

$$\begin{aligned} &f_1\{g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y\} \\ &+f_2\{g(X,\phi Z)\phi^2 Y - g(Y,\phi Z)\phi^2 X + 2g(X,\phi Y)\phi^2 Z\} \\ &= \frac{r}{2n(2n+1)}[g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X \\ &-g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y]. \end{aligned} \tag{21}$$

Taking inner product of the above equation with respect to W, we have

$$\begin{aligned} &f_1\{g(Y,Z)g(\phi X,W) - \eta(Y)\eta(Z)g(\phi X,W) - g(X,Z)g(\phi Y,W) \\ &+\eta(X)\eta(Z)g(\phi Y,W)\} + f_2\{g(X,\phi Z)g(\phi^2 Y,W) \\ &-g(Y,\phi Z)g(\phi^2 X,W) + 2g(X,\phi Y)g(\phi^2 Z,W)\} \\ &= \frac{r}{2n(2n+1)}[g(Y,Z)g(\phi X,W) - \eta(Y)\eta(Z)g(\phi X,W) \\ &-g(X,Z)g(\phi Y,W) + \eta(X)\eta(Z)g(\phi Y,W)]. \end{aligned} \tag{22}$$

On plugging $Y = Z = e_i$ in (22), we get

$$3f_2\, g(X,\phi W) = f_3(1-2n)g(X,\phi W), \tag{23}$$

which implies

$$f_3 = \frac{3f_2}{1-2n}. \tag{24}$$

Hence we can state the following:

Theorem 3.2 *A $(2n+1)$-dimensional generalized Sasakian space form is ϕ-concircular flat, then $f_3 = \frac{3f_2}{1-2n}$.*

4 Pseudo-concircular flat generalized Sasakian space forms

Definition 4.1 *A generalized Sasakian space form M^{2n+1} is said to be pseudo concircular flat if*

$$g(\tilde{C}(\phi X, Y)Z, \phi W) = 0, \ \forall\ X, Y, Z, W \in TM^{2n+1}. \tag{25}$$

Let us consider a $(2n+1)$-dimensional pseudo concircular flat generalized Sasakian space form. Then it follows from (25) that

$$g(\tilde{C}(\phi X, Y)Z, \phi W) = R(\phi X, Y, Z, \phi W) \tag{26}$$
$$-\frac{r}{2n(2n+1)}[g(Y,Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W)].$$

Let $e_1, e_2, \ldots, e_{2n+1}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (26) and taking summation over i, $(1 \leq i \leq 2n+1)$, then using (1), (2), (5) and (11), we obtain

$$S(\phi X, \phi W) = \frac{r}{2n+1} g(\phi X, \phi W). \tag{27}$$

Replacing X by ϕX and W by ϕW in (27), we get

$$S(X, W) = Ag(X, W) + B\eta(X)\eta(W), \tag{28}$$

where $A = \frac{r}{2n+1}$ and $B = \frac{r}{2n+1} - 2n(f_1 - f_3)$.
On contracting (28), we get

$$r = 2n(2n+1)(f_1 - f_3). \tag{29}$$

Thus we can state the following:

Theorem 4.2 *A $(2n+1)$- dimensional pseudo-concircular flat generalized Sasakian space form is an η-Einstein manifold with a scalar curvature $r = 2n(2n+1)(f_1 - f_3)$.*

5 Quasi-concircular flat generalized Sasakian space form

Definition 5.1 *A generalized Sasakian space form is said to be Quasi-concircular flat if it satisfies* $g(\tilde{C}(X,Y)Z,\phi W) = 0$ *for every vector fields* $X,Y,Z,W \in M^{2n+1}$,

Now consider a Quasi-concircular flat generalized Sasakian space form, then it can be easily seen that

$$g(\tilde{C}(X,Y)Z,\phi W) = R(X,Y,Z,\phi W) - \frac{r}{2n(2n+1)}[g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W)]. \quad (30)$$

Let $e_1, e_2, \ldots, e_{2n+1}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (30) and taking summation over i, $(1 \le i \le 2n+1)$, then using (1), (2), (5) and (11) gives

$$S(X,\phi W) = \frac{r}{2n+1} g(X,\phi W). \quad (31)$$

Replacing W by ϕW in (31) and then using (3), we get

$$S(X,W) = Ag(X,W) + B\eta(X)\eta(W), \quad (32)$$

where $A = \frac{r}{2n+1}$ and $B = 2n(f_1 - f_3) - \frac{r}{2n+1}$.
On contracting (32), we get

$$r = 2n(2n+1)(f_1 - f_3). \quad (33)$$

Thus we can state the following:

Theorem 5.2 *A* $(2n+1)$*- dimensional Quasi-concircular flat generalized Sasakian space form is an* η*-Einstein manifold with a scalar curvature* $r = 2n(2n+1)(f_1 - f_3)$.

6 Concircular pseudo-symmetric generalized Sasakian space forms

Definition 6.1 *A* $(2n+1)$*-dimensional generalized Sasakian space form* M^{2n+1} *is said to be concircular pseudo-symmetric if* $R \cdot \tilde{C} = L_{\tilde{C}}\, Q(g,\tilde{C})$. *i.e.,*

$$(R(X,Y)\tilde{C})(U,V)W = L_{\tilde{C}}[((X \wedge Y)\tilde{C})(U,V,W)], \quad (34)$$

where

$$(\xi \wedge_g X)Y = g(X,Y)\xi - g(\xi,Y)X. \quad (35)$$

Now consider concircular pseudo-symmetric generalized Sasakian space form. Then from (34), we have

$$(R(\xi,Y)\tilde{C})(U,V)W = L_{\tilde{C}}[((\xi\wedge Y)\tilde{C})(U,V,W)]. \tag{36}$$

Now left hand side of (36) gives

$$\begin{aligned}&(f_1-f_3)[g(\tilde{C}(U,V)W,Y)\xi-\eta(\tilde{C}(U,V)W)Y-\tilde{C}(g(Y,U)\xi\\&-\eta(U)Y,V)W-\tilde{C}(U,(g(Y,V)\xi-\eta(V)Y))W\\&-\tilde{C}(U,V)(g(Y,W)\xi-\eta(W)Y)].\end{aligned} \tag{37}$$

Similarly right hand side of (36) yields

$$\begin{aligned}&L_{\tilde{C}}[g(Y,\tilde{C}(U,V)W)\xi-\eta(\tilde{C}(U,V)W)Y-\tilde{C}((g(Y,U)\xi\\&-\eta(U)Y),V)W-\tilde{C}(U,(g(Y,V)\xi-\eta(V)Y))W\\&-\tilde{C}(U,V)(g(Y,W)\xi-\eta(W)Y)].\end{aligned} \tag{38}$$

By using (37) and (38) in (36) and then taking inner product with ξ, we get

$$\begin{aligned}&[L_{\tilde{C}}-(f_1-f_3)][\tilde{C}(U,V,W,Y)-\eta(\tilde{C}(U,V)W)\eta(Y)\\&+\eta(U)\eta(\tilde{C}(Y,V)W)+\eta(V)\eta(\tilde{C}(U,Y)W)\\&+\eta(W)\eta(\tilde{C}(U,V)Y)-g(Y,U)\eta(\tilde{C}(\xi,V)W)\\&-g(Y,V)\eta(\tilde{C}(U,\xi)W)-g(Y,W)\eta(\tilde{C}(U,V)\xi)]=0.\end{aligned} \tag{39}$$

By putting $U=Y$ in (39), we get either $L_{\tilde{C}}=f_1-f_3$ or

$$\begin{aligned}&[\tilde{C}(Y,V,W,Y)+\eta(V)\eta(\tilde{C}(Y,Y)W)+\eta(W)\eta(\tilde{C}(Y,V)Y)\\&-g(Y,Y)\eta(\tilde{C}(\xi,V)W)-g(Y,V)\eta(\tilde{C}(Y,\xi)W)\\&-g(Y,W)\eta(\tilde{C}(Y,V)\xi)]=0.\end{aligned} \tag{40}$$

On contracting Y in (40) and then using(1), (2), (4), (6), (11) and (17), we get

$$S(V,W)=Ag(V,W)+B\eta(V)\eta(W), \tag{41}$$

where $A=2n\left[f_1-f_3-\frac{r}{2n(2n+1)}\right]$ and $B=-4n\left[f_1-f_3-\frac{r}{2n(2n+1)}\right]$. This leads us to the following:

Theorem 6.2 *A $(2n+1)$-dimensional generalized Sasakian space form is concircular pseudo-symmetric, then either $L_{\tilde{C}}=f_1-f_3$ or the manifold is η-Einstein.*

7 ϕ-concircular semi symmetric generalized Sasakian space form

Definition 7.1 *A $(2n+1)$-dimensional generalized Sasakian space form is said to be ϕ-concircular semi-symmetric if*

$$\tilde{C}(X,Y)\cdot\phi = 0. \tag{42}$$

In view of (42), we have

$$(\tilde{C}(X,Y)\cdot\phi)Z = \tilde{C}(X,Y)\phi Z - \phi\tilde{C}(X,Y)Z = 0. \tag{43}$$

By using (1), (2), (3), (4), (9) in (43), we get

$$\begin{aligned}&\{f_1 - f_2 - \frac{r}{2n(2n+1)}\}[g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X \\ &+g(X,Z)\phi Y] = \{f_2 - f_3\}[g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi \\ &+\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y].\end{aligned} \tag{44}$$

On plugging $Y=\xi$ in (44) and then taking inner product with respect to ξ, we get

$$\{f_1 - f_3 - \frac{r}{2n(2n+1)}\}g(X,\phi Z) = 0. \tag{45}$$

Since $g(X,\phi Z)\neq 0$, we have

$$r = 2n(2n+1)(f_1 - f_3). \tag{46}$$

Hence we can state the following:

Theorem 7.2 *A $(2n+1)$-dimensional generalized Sasakian space form is ϕ-concircular semi-symmetric, then the scalar curvature is given by $r = 2n(2n+1)(f_1 - f_3)$.*

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Received: July 27, 2017