Mathematica Aeterna, Vol. 7, 2017, no. 3, 233 - 240

Eigensubspaces of endomorphisms of algebra of convergent power series

Aydin I. Shahbazov

Inst. of Math. and Mech. of NAS of Azerb.
Az. Ped. St.University
aydinshahbazov@yahoo.com

Dashqin A. Seyidov

Nakhchivan State University
dashqinseyidov@gmail.com

Abstract

In [5] was considered the eigenvalues and eigensubspaces of endomorphisms (which are induced by the selfmappings with Denjoy-Wolff type fixed points) of algebra of convergent power series Σ_n of n variables $z = (z_1, \ldots, z_n)$ and for the algebras Σ_2 was determined the eigenvalues (also, described their corresponding eigensubspaces) of endomorphisms in the resonancing cases. In this work we continue this problem and we determine the eigenvalues (also, describe their corresponding eigensubspaces) of such endomorphisms of algfbras Σ_2 in the all cases.

Mathematics Subject Classification: 47B38

Keywords: resonancing and nonresonancing eigenvalues, resonancing monom, eigensubspace, endomorphism, Denjoy-Wolff type fixed point, uniform algebra.

1 Introduction

In [2] Kamowitz considered the weighted composition operator T on the disc-algebra (i.e. the algebra of continuous functions on the closed unit disc and analytic in the interior of its) and was determined its spectrum in the case when T is compact. In [3] we have more generally results inclusion multidimensional cases. In [3] was considered the weighted composition operators on uniform spaces of analytic functions, which induced by the compressly mappings on the bounded domains $D \subset C^n$ $(n \geq 1)$ and was determined its spectrum. Another words, if D is a bounded domain and $\varphi : \overline{D} \to D$ is holomorphic mapping (where $\overline{D}$ denote closure of D), then in [3] was considered the operators of the form $T : X \to X,\ f \to u\, f \circ \varphi$, for every $f \in X$, where $u \in X$ is fixed function and X is Banach-A(D) module, which is uniform subspace of space of holomorphic functions on D equipped with uniform topology. It is well known the mapping φ has a unique fixed point in D. In [3] was shown the spectrum of operator T is equal to semigroup induced by eigenvalues of linear part of φ at the fixed point. Since these operators are compacts, then every eigensubspace corresponding to nonzero eigenvalue has finite dimensions. But from method of [3] we know about dimensions of eigensubspaces, if only case when differential of mapping φ at the fixed point has differently, nonzero and multiplicativly independent eigenvalues (and in this case corresponding eigensubspace has dimension 1). Since in this case between eigenvalues and eigensubspaces of operator T and eigenvalues and eigensubspaces of endomorphism of algebra of formal (or convergent) series there are bijective mapping (see [4]), so we begin investigate last problems. In [5] avoid the results of [3] was calculated directly the spectrum and was discribed eigensubspaces of operator T induced by the selfmappings with Denjoy-Wolff type fixed point in the resonancing case, when $n = 2$. Without loss of generality, we may assume, as so as [5], weighted function u is identity, and domain of φ, which induced the weighted endomorphism T contains the origin of coordinate and it is fixed point for mapping φ (i.e., we will consider the operator T : $f \to f \circ \Phi$ on the algebra $\sum_2$).

Investigation of spectral properties (for example, spectrum, eigenvalues, eigensubspaces and so) of endomorphisms, also weighted endomorphisms on different algebras (for example, on the uniform algebras, especially on the function algebras with analytic structure , etc), usually leads to investigation these problems on the algebras formally convergent power series (instance , in the case algebra of analytic functions, we have the algebra of germs of functions at the fixed points, etc). Moreover, in many cases studying some algebraic and spectral properties of endomorphisms, or weighted endomorphisms induced by compression mappings (for example, see [3]), or more generally, by the mappings which have fixed points, in some sense(for example, in the Denjoy-Wolff sense fixed point, and so) on the function algebras with analytic structure, again leads to studying endomorphisms of above mentioned algebras. Especially, on the uniform algebras spectrum of the compact, or quazi-compact weighted endomorphisms described by the eigennumbers of linear part of endomorphism at the origin, which modules less than 1 (see [3]). So, in this work, so as [5] we will assume that modules of eigennumbers of the linear part of mapping (which induced the given endomorphism) on initial point of coordinate system less than 1.

Let $\sum_n$ be the algebra of convergent power series of $n \geq 1$ variables

$z = (z_1, \ldots, z_n)$. In [5] was considered In this algebra eigenvalues of endomorphism generated by mapping Φ which modules of all eigenvalues of its linear part Φ_1 at the origin less than 1, nonzero, differently and nonresonancing (if, $\alpha_1, \ldots, \alpha_n$ are eigenvalues of Φ_1 , then α_s is called resonancing eigenvalue, if $\alpha_s = \alpha^m = \alpha_1^{m_1} \ldots\ \alpha_n^{m_n}$, where, $m_i \geq 0$, $\sum_{i=1}^n m_i \geq 2$; for any resonancing eigenvalue $\alpha_s = \alpha^m$ corresponding resonancing vector-monom $z^m e_s$, where, e_s is basic vector and $z^m = z_1^{m_1} \ldots z_n^{m_n}$; if, between eigenvalues $\alpha_1, \ldots, \alpha_n$ there is resonancing conditions, then the endomorphism is called resonancing endomorphism, otherwise is called nonresonancig endomorphism). It is clear that in this case by Puancare's theorem (see [1]) we may by diffeomorphic transformation of coordinates the mapping Φ reduced to its linear part Φ_1 , which has diagonal form. Let Φ_1 has a form $\Phi_1 = diag\ (\alpha_1, \ldots, \alpha_n)$. In this case in [5] was proved next theorem:

Theorem 1.1 *If modules of eigennumbers $\alpha_1, \ldots, \alpha_n$ of the linear part of mapping Φ which generated the endomorphism $T : \Sigma_n \to \Sigma_n$ are less than 1 and nonzero,nonresonansing, differently, then eigenvalues of T have the form $\lambda_{k=}\alpha_1^{k_1} \ldots\ \alpha_n^{k_n}$, where k=($k_1, \ldots k_n$), $k_i \in Z_+$, $i = 1, \ \ldots\ , n$, and corresponding eigensubspaces up to diffeomorphism are generated by the functions $f_k = z^k$ (consequently, all eigensubspaces are one dimensional).*

In [5] further, was considered the algebra Σ_2 of series (formal or convergent series) of the form

$$\sum_{n,m} a_{n,m} x^n y^m \ ,$$

and endomorphism T of this space induced by formal series Φ, which module of eigenvalues of linear part of Φ are less than 1, i.e. was considered the operator of the form: $T : \Sigma_2 \ \to \Sigma_2, \quad f \to f \circ \Phi \quad (f \in \Sigma_2)$ where eigenvalues α_1, α_2 of the linear part of Σ holds: $a_i : \quad 0 < |a_i| < 1 \qquad (i = 1, 2)$. In the resonancing case in [5] corresponding with the resonancing monoms and for the without resonancing monoms wase proved next theorems:

Theorem 1.2 *In the resonancing case with the resonancing monoms every eigenvalue of endomorphism $T : \Sigma_2 \to \Sigma_2$ has the form $\lambda_q = \alpha^q$ (where q is nonnegative whole number) and corresponding eigenfunction has the form $f_q(x, y) = y^q$ (or has the form $f_q(x, y) = x^q$). Consequently, corresponding eigensubspaces are one-dimensional.*

Theorem 1.3 *In the resonancing cases without resonancing monoms every eigenvalue of endomorphism $T : \Sigma_2 \to \Sigma_2$ has the form $\lambda_q = \alpha^q$ (where q is nonnegative whole number) and corresponding eigenfunctions have the forms $f\ (x, y) = \sum_{k=0}^{\left[\frac{q}{m}\right]} a_{k,q-mk} x^k y^{q-mk}$ (or has the form $f\ (x, y) = \sum_{k=0}^{\left[\frac{q}{m}\right]} a_{k,q-mk} y^k x^{q-mk}$).*

Consequently, corresponding eigensubspaces are ($\left[\frac{q}{m}\right]+1$)- dimensional, where m is an order of resonancing conditions and $[r]$ denote a whole part of r.

So, continue this problem we will consider the rest cases.

2 The case, when between eigennumbers of the linear part of mapping there are not resononcing conditions

Now we consider the case, when eigennumbers α_1, α_2 of the linear part of mapping Φ, which induced endomorphim $T : \Sigma_2 \to \Sigma_2, \quad f \to f \circ \Phi \quad (f \in \Sigma_2)$ are differently and between them there are not resonancing conditions. We will find eigenvalues of endomorphism T and we will calculate the dimensions of corresponding eigensubspaces. By using Poincare's theorem (see [1]) we assume that, Φ has the form $\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ where $0 < |\alpha_1|, \; |\alpha_2| < 1$. Put $f(x,y) = \sum_{k,l} a_{k,l} x^k y^l$ and we consider eigenvalues problems:

$$(Tf)(x,y) = \sum a_{k,l} (\alpha_1 x)^k (\alpha_2 y)^l = \lambda f = \lambda \sum a_{k,l} x^k y^l.$$

From this for any $k, l \geq 0$ we get $a_{k,l}\left({\alpha_1}^k {\alpha_2}^l - \lambda\right) = 0$. If, for some $a_{k_0,l_0} \neq 0$ then $\lambda = {\alpha_1}^{k_0} {\alpha_2}^{l_0}$. If we assume that for the another $a_{k_1,l_1} \neq 0$, where $\{k_1, l_1\} \neq \{k_0, l_0\}$, so we have that $\lambda = {\alpha_1}^{k_1} {\alpha_2}^{l_1}$ and from this we get ${\alpha_1}^{k_0} {\alpha_2}^{l_0} = {\alpha_1}^{k_1} {\alpha_2}^{l_1}$, i.e. ${\alpha_1}^{k_1 - k_0} {\alpha_2}^{l_1 - l_0} = 1$. In this case we have next conditions.
a) When between eigenvalues α_1, α_2 there are not multiplicative depending relations (i.e., the eigenvalues α_1, α_2 such that, for any pairs $\{k_0, l_0\} \neq \{0, 0\}$ we have ${\alpha_1}^{k_0} {\alpha_2}^{l_0} \neq 1$) the next theorem is clear:

Theorem 2.1 *If, between eigennumbers α_1, α_2 there are not resonancing and multiplicative conditions, then any eigenvalue of endomorphism $T : \Sigma_2 \to \Sigma_2, \quad f \to f \circ \Phi \quad (f \in \Sigma_2)$ has the form $\lambda_{(k,l)} = {\alpha_1}^k {\alpha_2}^l$ and for this eigenvalue corresponding unique (up to multiplier) eigenfunction $f(x,y) = x^k y^l$; so we have $dim E_T(\lambda_{(k,l)}) = dim E_T\left({\alpha_1}^k {\alpha_2}^l\right) = 1$ (if, λ is an eigenvalue for the endomorphism T, then by $E_T(\lambda)$ we denote the eigensubspace which corresponding to eigenvalue λ).*

b) Now we will consider the second case i.e., the case, when between eigenvnumbers there are multiplicative relations i.e., the eigennumbers α_1, α_2 are such that between these there are multiplicative depending conditions. Then there is some $\{k_0, l_0\} \neq \{0, 0\}$, such that ${\alpha_1}^{k_0} {\alpha_2}^{l_0} = 1$ (it is clear that in this case both k_0 and l_0 are not zero, because $0 < |\alpha_1|, \ |\alpha_2| < 1$). Therefore we consider the equation

$$\alpha_1{}^a \alpha_2{}^b = 1, \quad 0 < |\alpha_1|, \ |\alpha_2| < 1, \tag{$*$}$$

and look for it's integer solutions. Suppose, that the eigennumbers α_1 and α_2 have the forms $\alpha_1 = \rho_1 e^{2\pi i t_1}, \ \alpha_2 = \rho_2 e^{2\pi i t_2}$ where $0 \leq t_1, \quad t_2 < 1$, then we have

$$\begin{cases} \rho_1^a \rho_2^b = 1 \\ at_1 + bt_2 \in Z, \end{cases}$$

$$alog\rho_1 + blog\rho_2 = 0, \tag{1}$$

$$at_1 + bt_2 \in Z, \tag{2}$$

where Z is the set of whole numbers. Since we look for the integer nonzero solutions $a \neq 0, b \neq 0$, so from (1) we conclude, that $\frac{log\rho_2}{log\rho_1}$ is rational number. Thus we get the necessary condition for the solution (*):

$$log\rho_2 / log\rho_1 \in Q, \quad \text{where } Q \quad \text{is the set of rational numbers.}$$

We assume that the above conditions holds, i.e., we have next representation:

$$log\rho_2 / log\rho_1 = \frac{m_2}{m_1}, m_1, \ m_2 \in Z, \ (|m_1|, \ |m_2|) = 1,$$

where the bracket (,) denote the greatest common divisor (in fact, since $0 < \rho_1, \ \rho_2 < 1$, so $log\rho_1, log\rho_2 < 0$, i.e., $log\rho_2/log\rho_1 > 0$, therefore we may assume that, $m_1, \ m_2 > 0$). With these conditions numbers $m_1, \ m_2$ are defined unambiguously with respect to $\rho_1, \ \rho_2$. From (1) we get $a = -blog\rho_2/log\rho_1 = -\frac{m_2}{m_1}b$; then from (2) we have:

$$-t_1 \frac{m_2}{m_1} b + t_2 b \in Z, \quad \text{i.e., } b(t_2 - t_1 \frac{m_2}{m_1}) \ \in Z. \tag{2'}$$

Since b must be integer and nonzero, then we have the number $t_2 - t_1 \frac{m_2}{m_1}$ is rational:

$$t_2 - t_1 \frac{m_2}{m_1} \in Q \tag{3}$$

This is necessary condition for the existence solutions. We assume that the condition (3) holds and we have the next representation:

$$t_2 - t_1 \frac{m_2}{m_1} = \frac{n_2}{n_1} \ , n_2, n_1 \in Z, \ (|n_2|, \ n_1) = 1, \ n_1 > 0.$$

It is clear by these conditions numbers $n_1, \ n_2$ are defined unambiguously (with respect to $t_1, \quad t_2$, $\rho_1, \ \rho_2$). Since, $log\rho_2 = \frac{m_2}{m_1} log\rho_1$, so the equation (1) has the form

$$alog\rho_1 + b\frac{m_2}{m_1} log\rho_1 = 0 \quad (log\rho_1 \neq 0)$$

i.e.,

$$m_1 a + m_2 b = 0 \tag{4}$$

Take into account of $m_1,\ m_2$ are mutual prime numbers, so, we get common solution for equation (4) by the next form:

$$a = km_2, \quad b = -km_1, \quad k \in Z$$

From $(2')$ and from that $t_2 - t_1\frac{m_2}{m_1} = \frac{n_2}{n_1}$, $b = -km_1$, we conclude the number $(-km_1 \cdot \frac{n_2}{n_1})$ is integer. Since $(n_2, n_1) = 1$, so the number km_1 must be divided by n_1. Put $d = (m_1, n_1)$; then we have

$$m_1 = pd;\ n_1 = qd; (p,q) = 1; km_1 = kpd;\ \frac{km_1}{n_1} = \frac{kpd}{qd} = \frac{kp}{q}$$

and they are integers. Since $(p,q) = 1$, so we get $k = qs$, where $s \in Z$. Thus, the common solutions of the equation (*) on the ring of integer numbers we get in the next forms:

$\left.\begin{array}{l} a = sqm_2 \\ b = -sqm_1 \end{array}\right\}$ where $s \in Z,\ \ s \neq 0$ (we look for nonzero solutions).

(It is clear , for these numbers a and b are hold next condition:

$$a log\rho_1 + b log\rho_2 = 0$$

$$at_1 + at_2 \in Z$$

which is equivalent to equation (*)).
Thus, we give next theorem:

Theorem 2.2 *If, between eigennumbers α_1, α_2 of the linear part of mapping Φ there is not resonancing conditions, but they are multiplicative depending, then every eigenvalue of the operator $T : \Sigma_2 \to \Sigma_2, \quad f \to f \circ \Phi \quad (f \in \Sigma_2)$ has the form $\lambda_{(k,l)} = {\alpha_1}^k {\alpha_2}^l$ for some $(k,l) \in Z_+ \times Z_+$ and corresponding eigensubspace consists of polynoms:*

$$f(x,y) = \sum_{\substack{S \in Z \\ k + sqm_2 \geq 0 \\ l - sqm_1 \geq 0}} a_s x^{k+sqm_2} y^{l-sqm_1}$$

and therefore corresponding eigensubspace is finite dimensional, and:

$$dim E_T\left({\alpha_1}^k {\alpha_2}^l\right) = |Z \cap [-kd/n_1 m_2, \quad ld/n_1 m_1]|$$

(where the symbol $|A|$- defined power of the set A).

Remark 2.3 *In the nonrezonancing case, when eigennumberss of linear part of mapping* Φ *are coincides, i.e., the linear part of mapping* Φ *has the normal form* $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$, *then it easy showed that this case is similarly to the resonancing case with the rezonancing monoms (see Theorem 1.2)) , and for eigenvalues of the forms* $\lambda = \alpha^q$ *(where* q *is nonnegative whole number) corresponding are monoms:* $f_q(x, y) = y^q$ *(or* $f_q(x, y) = x^q$*). Thus in this case for any nonnegative whole number* q *we get* $dimE_T(\alpha^q) = 1$.

3 The case, when a linear part of inducing mapping has a zero eigennumber

Finally we consider the case when one of the eigennumbers α_i $(i = 1, 2)$ of the linear part of mapping Φ is zero (it is clear, this is resonancing case). If, one of the α_i is nonzero, then without loss of generality we can reduce the mapping Φ to the form $\Phi = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$, i.e., we have $\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ o \end{pmatrix}$ .Then for eigenvalue problem we get:

$$(Tf)(x, y) = \sum_{k,l} a_{k,l}(\alpha x)^k = \sum_{k=0}^{\infty} a_{k,0}\alpha^k x^k = \lambda \sum_{k,l} a_{k,l}x^k y^l = \sum_{k=0}^{\infty} x^k (\sum_{l=0}^{\infty} \lambda a_{k,l}y^l)$$

For any k we get:

$$a_{k,0}\alpha^k = \sum_{l=0}^{\infty} \lambda a_{k,l}y^l = \lambda a_{k,0} + \sum_{l=1}^{\infty} \lambda a_{k,l}y^l$$

i.e.

$$a_{k,0}(-\alpha^k + \lambda) + \sum_{l=1}^{\infty} \lambda a_{k,l}y^l = 0$$

and form this we have: $a_{k,0}(-\alpha^k + \lambda) = 0$ for any k and $\lambda a_{k,l} = 0$ for any k, and any $l \geq 1$.

So, we have two cases:

c_1) The case when there is exists nonzero eigenfunction.

If for some k_0 we have $a_{k_{0,0}} \neq 0$, then $\lambda = \alpha^{k_0} \neq 0$ and $a_{k,l} = 0$ for any k, and for any $l \geq 1$. Therefore, we get $f(x, y) = \sum_{k=0}^{\infty} c_k x^k$, and easy be showed, that indeedly $c_{k_0} \neq 0$, and $c_k = 0$, for any $k \neq k_0$. Consequently, eigenvalues of operator T have the forms $\lambda_k = \alpha^k$ and according eigenfunctions consists of the monoms x^k (or y^k), i.e. $dimE_T\left(\alpha^k\right) = 1$.

c_2) The case when there is not nonzero eigenfunction.

If for any $k:\ a_{k,0}=0$, then we get, either $\lambda=0$ and $f(x,y)=\sum_{l=0}^{\infty} c_l y^l$ (i.e., the eigenvalue is zero and corresponding egensubspace is infinite dimensional eigensubspace), or $\lambda\neq 0$ and $a_{k,l}=0$ for any k and any $l\geq 1$, but then we have $f\equiv 0$, i.e., there are not eigenfunctions.

Theorem 3.1 *If, one of eigennumbers of the linear part of inducing mapping is zero, then,every nonzero eigenvalue of the operator* $T:\Sigma_2\to\Sigma_2,\quad f\to f\circ\Phi\quad (f\in\Sigma_2)$ *has the form* $\lambda_k=\alpha^k$ *for some* $k\in Z_+$ *and corresponding eigenfunctions consists of the monoms* x^k *(or* y^k *), i.e.,* $dimE_T\left(\alpha^k\right)=1$.

Remark 3.2 *The case when both eigenvalues of linear part of mapping* Φ *are equal to zero (i.e.* $\alpha_1=\alpha_2=0$ *then* $\Phi(\mathrm{x},\mathrm{y})=0(|\mathrm{x}|^2+|\mathrm{y}|^2)$*), it is clear, that there is not exists eigenunction (except, the case* $\lambda=1$ *and* $f\equiv const$*).*

References

[1] V.I. Arnold, Complementary chapters of theory of ordinary differential equations, Nauka, 1978.

[2] H. Kamowitz, Compact operator of the form, Pacific Jour. of Math., 1979, vol.80, No 1, pp.205-211.

[3] A.I. Shahbazov, Spectrum of compact operator of weighted composition in certain Banach spaces of holomorphic functions, Jour. Sov. Math., 1990, vol.48, No 6, pp.696-701.

[4] A.I. Shahbazov, R.A. Imamquliyev, Compact weighted composition operators on the space of holomorphic functions, Trans. Nas of Azerb., 2007, vol.27, No 1, pp.123-128.

[5] A.I. Shahbazov, D.A. Seyidov, Eigensubspaces of resonancing endomorphisms of algebra of convergent power series, Caspian Jour. of Appl. Mathematics, Ecology and Economics, V.3, No 2, 2015, pp.77-84.

Received: August 10, 2017