Mathematica Aeterna, Vol. 7, 2017, no. 3, 261 - 267

A Note on Implicit Fractional Differential Equations

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Abstract

To correct an error in [1], this note deals with the basic theory of nonlinear implicit fractional differential equations involving Caputo fractional derivative. By an alternative way different with [1], we investigate the existence, interval of existence of solutions and uniqueness of an implicit fractional differential equation.

Mathematics Subject Classification: 26A33; 34A12

Keywords: : Implicit fractional differential equations. Caputo fractional derivative. Fixed point theorem. Existence and uniqueness

1 Introduction

Recently in [1], the authors have studied the nonlinear implicit fractional differential equations as follows:

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t), {}^c D^\alpha x(t)) \\ x(0) = x_0, \end{cases} \tag{1.1}$$

In Theorem 3.1 of [1], they asserted that the initial value problem for IFDE (1.1) is equivalent to nonlinear fractional volterra integro-differential equation :

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), {}^c D^\alpha x(s)) ds. \ \ t \in [0, T].$$

In Theorem 3.2 of [1], let $B = C([0,\chi], R), U = \{x \in B : x(0) = x_0, \|x - x_0\| \leq \xi$. An operator A is defined on U by:

$$Ax(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), {}^c D^\alpha x(s)) ds. \quad t \in [0.\chi].$$

They proved that A maps U into U.

However, we find that the functions $x(t)$ in U isn't always differentiable. If $x(t)$ isn't differentiable, then $\frac{d}{dt}x(t)$ and

$${}^c D^\alpha x(s) = \frac{1}{\Gamma(1-\alpha)} \int_0^s (s-t)^{-\alpha} \frac{d}{dt} x(t) dt,$$

are meaningless. Therefore, the proof $A : U \to U$ is incorrect. To correct this error, in this note, we investigate the existence, interval of existence and uniqueness of the IFDE(1.1). By an alternative way, we turn the problem (1.1) into an equivalent equation, then, the solvability of this equivalent equation implies the existence of solution to problem (1.1).

2 Preliminary Notes

Definition 2.1. *In this section, some basic definitions and results of fractional calculus [3,6,7,15] which are used throughout this paper are presented.*

Definition 2.1 *A real valued function $f(t)(t > 0)$ is said to be in space $C_\mu(\mu \in R)$ if there exists a real number $p(p > \mu)$ such that $f(t) = t^p g(t)$, where $g \in C[0, \infty)$.*

Definition 2.2 *A real function $f(t)(t > 0)$ is said to be in the space $C_\mu^n, n \in N \cup \{0\}$, if $f^{(n)} \in C_\mu$.*

Definition 2.3 *let $f \in C_\mu(u \geq -1)$, then the (left-sided) Riemann-Liouville fractional integral of order $\alpha > 0$ of the function f is given by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, t > 0 \quad and \quad I^0 f(t) = f(t).$$

where $\Gamma(\cdot)$ is Euler's Gamma function .

Definition 2.4 *The (left-sided) Caputo fractional derivative of order $\alpha > 0$ of the function $f \in C_{-1}^n (n \in N \cup \{0\})$, is given by*

$${}^c D^\alpha f(t) = \begin{cases} f^{(n)}(t), if \quad \alpha = n, \\ I^{n-\alpha} f^{(n)}(t), if \quad n-1 < \alpha < n, n \in N \end{cases}$$

Note that $I^\alpha \; {}^c D^\alpha f(t) = f(t) - \Sigma_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} t^k, n-1 < \alpha \leq n, n \in N$

Definition 2.5 *The (left-sided) Riemann-Liouville fractional derivative of order* $\alpha > 0$ *of the function* $f \in C^n_{-1}(n \in N \cup \{0\})$*, is given by*

$$D^\alpha f(t) = \frac{d^n}{dt^n} I^{n-\alpha} f(t), n-1 < \alpha \le n, n \in N$$

Following relation holds between Caputo and Riemann-Liouville fractional derivatives:

$$^cD^\alpha f(t) = D^\alpha (f(t) - \Sigma_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} t^k), n-1 < \alpha \le n, n \in N$$

Definition 2.6 *The function* $E_\alpha(\alpha > 0)$ *defined by* $E_\alpha(z) = \Sigma_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$ *, is called Mittag-Leffler function of order* α.

Lemma 2.1 *Let* $\alpha, \beta \in [0, \infty)$*. Then*

$$\int_0^t s^{\alpha-1}(t-s)^{\beta-1} ds = t^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The following generalization of Gronwalls lemma for singular kernels plays an important role in obtaining our main results .

Lemma 2.2*([14],Lemma 7.1.1) Let* $v : [0, T] \to [0, \infty)$ *be a real function and* $\omega(\cdot)$ *is a nonnegative, locally, integrable function on* $[0, T]$*. Assume that there is a constant* $a > 0$ *such that for* $0 < \alpha \le 1$

$$v(t) \le w(t) + a \int_0^t (t-s)^{-\alpha} v(s) ds$$

Then, there exist a constant $K = K(\alpha)$ *such that*

$$v(t) \le w(t) + Ka \int_0^t (t-s)^{-\alpha} \omega(s) ds$$

for every $t \in [0, T]$*. The proof of Lemma 2.2 is given in [13].*

3 Main Results

Existence Results and Interval of Existence

In this section, we give the equivalence of the initial value problem for IFDE(1.1) and prove the existence of solutions.

Lemma3.1[1]**.** If the function $f : [0, T] \times R^2 \to R$ is continuous , then the initial value problem for IFDE(1.1) is equivalent to nonlinear fractional Volterra integro-differential equation:

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), {}^c D^\alpha x(s)) ds. \quad t \in [0, T]$$

This lemma gives the equivalence of the initial value problem for IFDE(1.1) to an integro-differential equation .

Theorem3.2. Let $T > 0$. Assume $f : [0,T] \times R^2 \to R$ satisfies the following condition
(H_1): There exists $K_1 \in R^+, K_2 \in (0,1)$ be constants such that :

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \leq K_1|x - \bar{x}| + K_2|y - \bar{y}|$$

Let

$$\chi < \min\{T, (\frac{(1-K_2)\cdot\Gamma(\alpha+1)}{K_1})^{\frac{1}{\alpha}}\}$$

Then the IFDE(1.1) has a unique solution $x : [0,\chi] \to R$.
Proof : Let

$${}^c D^\alpha x(t) = z(t) \ , x(t)|_{t=0} = x_0.$$

then, by Lemma 3.1,

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds. \quad t \in [0, T]$$

that is $x(t) = x_0 + I^\alpha z(t)$. Moreover, the initial value problem for IFDE(1.1) is equivalent to the following equation:

$$z(t) = f(t, x_0 + I^\alpha z(t), z(t)) \tag{3.1}$$

Let $z(t) \in C[0,\chi]$. Define the mapping A on $C[0,\chi]$ as follows

$$Az(t) = f(t, x_0 + I^\alpha z(t), z(t))$$

Then A maps $C[0,\chi]$ into $C[0,\chi]$. Let $z, \bar{z} \in C[0,\chi]$, then

$$|Az(t) - A\bar{z}(t)| = |f(t, x_0 + I^\alpha z(t), z(t)) - f(t, x_0 + I^\alpha \bar{z}(t), \bar{z}(t))|$$

Assumption (H_1) implies :

$$\begin{aligned} |Az(t) - A\bar{z}(t)| &\leq K_1 I^\alpha |z(t) - \bar{z}(t)| + K_2|z(t) - \bar{z}(t)| \\ &= K_1 \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - \bar{z}(s)| ds + K_2 |z(t) - \bar{z}(t)| \\ &\leq K_1 \frac{t^\alpha}{\Gamma(\alpha+1)} \max_{t\in[0,\chi]} |z(s) - \bar{z}(s)| + K_2 \max_{t\in[0,\chi]} |z(t) - \bar{z}(t)| \\ &= (K_1 \frac{t^\alpha}{\Gamma(\alpha+1)} + K_2) \max_{t\in[0,\chi]} |z(t) - \bar{z}(t)| \end{aligned}$$

Since $t \in [0, \chi]$, hence

$$\|Az(t) - A\bar{z}(t)\| \leq \beta\|z(t) - \bar{z}(t)\|. \quad 0 < \beta < 1$$

That is to say the operator A is a contraction mapping in $C[0, \chi]$, and hence A has a unique fixed point $z : [0, \chi] \to R$, which is the solution of IFDE (3.1). Therefore, the IFDE (1.1) has a unique solution x, where $x = x_0 + I^\alpha z$.

Estimates on the solutions

Theorem 3.3. Assume that : $f : [0, T] \times R^2 \to R$ satisfies (H_1). If $x(t), t \in [0, T]$ is a solution of IFDE (1.1), then

$$|x(t)| \leq |x_0| + T^\alpha \frac{(1 - K_2)\Gamma(\alpha + 1) + K_1 \cdot K \cdot T^\alpha}{(1 - K_2)^2(\Gamma(\alpha + 1))^2}(K_1|x_0| + Q)$$

where $Q = \sup_{t\in[0,T]} |f(t, 0, 0)|$ and $K \in R^+$ is a constant .

Proof Let

$$^cD^\alpha x(t) = z(t) \ , x(t)|_{t=0} = x_0.$$

By Lemma 3.1 , $x(t) = x_0 + I^\alpha z(t)$. Therefore

$$x(t) \leq |x_0| + I^\alpha|z(t)|$$

By hypothesis (H_1), for any $t \in [0, T]$

$$\begin{aligned}|z(t)| &= |f(t, x_0 + I^\alpha z(t), z(t))| \\ &\leq |f(t, x_0 + I^\alpha z(t), z(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq K_1|x_0 + I^\alpha z(t)| + K_2|z(t)| + |f(t, 0, 0)| \\ &\leq K_1|x_0| + K_1|I^\alpha z(t)| + K_2|z(t)| + |f(t, 0, 0)|.\end{aligned}$$

Hence

$$\begin{aligned}(1 - K_2)|z(t)| &\leq K_1|x_0| + |f(t, 0, 0)| + K_1 I^\alpha|z(t)| \\ &\leq K_1|x_0| + |f(t, 0, 0)| + K_1\frac{1}{1 - K_2}I^\alpha\{(1 - K_2)|z(t)|\}\end{aligned}$$

Let $Q = \sup_{t\in[0,T]} |f(t, 0, 0)|$. By Lemma 2.2, there is a constant K depending on α such that

$$\begin{aligned}(1 - K_2)|z(t)| &\leq K_1|x_0| + Q + K_1\frac{K}{(1 - K_2)\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}(K_1|x_0| + Q)ds \\ &\leq K_1|x_0| + Q + \frac{K_1 \cdot K \cdot T^\alpha}{(1 - K_2)\Gamma(\alpha + 1)}(K_1|x_0| + Q)\end{aligned}$$

Hence

$$|z(t)| \leq \frac{(1-K_2)\Gamma(\alpha+1)+K_1\cdot K\cdot T^{\alpha}}{(1-K_2)^2\Gamma(\alpha+1)}(K_1|x_0|+Q)$$

Therefore

$$\begin{aligned}|x(t)| &\leq |x_0| + I^{\alpha}|z(t)|\\ &\leq |x_0| + T^{\alpha}\frac{(1-K_2)\Gamma(\alpha+1)+K_1\cdot K\cdot T^{\alpha}}{(1-K_2)^2(\Gamma(\alpha+1))^2}(K_1|x_0|+Q)\end{aligned}$$

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Received: August 30, 2017