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THE FRACTIONAL INTEGRAL OPERATOR AND I-FUNCTION

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ABSTRACT: The object of the present paper is to drive the certain expansion theorems, which results from interconnected Laplace Transform with Weyl fractional integral operator involving I-function. On account of general nature of this function a number of results involving special function can be obtained by specializing the parameters [2].

Key words: Laplace Transform, Mellin Transform, Weyl fractional Integral operator, I-function.

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1. INTRODUCTION: The Weyl fractional integral is defined in the following form

$$W^{\mu}\{f(t):p\} = \frac{1}{\Gamma(\mu)}\int_{p}^{\infty}(t-p)^{\mu-1} f(t)\,dt \; ; \qquad \text{where } \mathrm{Re}(\mu) > 0 \qquad \ldots 1.1$$

The Laplace transform of f (t) is denoted by L [f (t)] is defined as

$$L[f(t):p] = \int_{0}^{\infty} e^{-pt} f(t)\,dt = F(p) \qquad \ldots 1.2$$

Here we stabilized a formula exhibiting a relationship between (1.1) and (1.2) which provides the more effective tools and allow the straight forward derivation of the

Weyl fractional integral operators associated with Saxena`s I-function, Fox`s H-function and Meijer`s G-function.

Expansion theorem involving double series have been established earlier by Jain and Pathan [6, 2001; 7, 2004].

2. I-FUNCTION: In general the Saxena's I-function [10, 1982] defined with the following integral on the complex plane:

$$I(t) = I^{m,n}_{p_i,q_i,r}[t] = I^{m,n}_{p_i,q_i,r}\left[t \left| \begin{matrix} \{(a_j,\alpha_j)_{1,n} & \ldots & (a_{ji},\alpha_{ji})_{n+1,p_i}\} \\ \{(b_j,\beta_j)_{1,m} & \ldots & (b_{ji},\beta_{ji})_{n+1,q_i}\} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi w}\int_L \left\{ \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}} \right\} t^s\, ds \qquad \ldots 2.1$$

Where $w=\sqrt{(-1)}$ p_i (i = 1,2…r), q_i (i = 1,2…r), m, n are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i$ (i = 1,2,…r), r is finite $\alpha_j,\beta_j \alpha_{ji},\beta_{ji}$ are real positive and $a_j,b_j\ a_{ji},b_{ji}$ all are complex numbers such that $a_j(b_h+v) \ne \beta_h(a_i-1-k)$ for v, k = 0,1,2… ;h = 1,2, …,m; i = 1,2, … n.

L is the contour running from σ - i∞ to σ + i∞ (σ is real) in the complex s plane such that

$s=(a_j-1-v)/a_j$ j = 1, 2, n; v = 0, 1, 2 …

$s=(b_j+v)/\beta_j$ j = 1, 2, m; v = 0, 1, 2 …

lie to the left hand and right hand sides of L respectively.

3. LAPLACE TRANSFORM OF I-FUNCTION: By the definition of Laplace transform [8, 1995; 9, 2012]. we get $L[I(t):p] = \int_0^\infty e^{-pt} I(t)\, dt$

$$= \int_0^\infty e^{-pt} \frac{1}{2\pi w} \int_L \left\{ \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} s) \right\}} \right\} t^s \, ds \; dt$$

By changing the order of integration

$$= \frac{1}{2\pi w} \left[\int_L \left\{ \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} s) \right\}} \right\} \int_0^\infty e^{-pt} t^s \, dt \right] ds$$

$$= \frac{1}{2\pi w} \int_L \left\{ \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s) \Gamma(s+1)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} s) \right\} p^{s+1}} \right\} ds$$

$$Q(p) = \frac{1}{2\pi w} \int_L \frac{\theta(s) \Gamma(s+1)}{p^{s+1}} ds = \hat{I}(p) \qquad \ldots 3.1$$

Where
$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} s) \right\}} \qquad \ldots 3.2$$

Now we shall establish a theorem involving Laplace Transform of Saxena's I-function

4. Theorem: Consider the Integral

$$Q(p) = \frac{1}{2\pi w} \int_L \frac{\theta(s) \Gamma(s+1) t^s}{p^{s+1}} ds \qquad \ldots 4.1$$

Where $\theta(s)$ is given by (3.2) Then under the assumption of absolute convergence

$$W^{\mu}\{t^{-\lambda}I(t):p\}=p^{\mu-\lambda-1}\frac{1}{2\pi w}\int_L\frac{\theta(s)\Gamma(1-\mu+\lambda+s)}{\Gamma(1+\lambda+s)p^s}ds$$

$$=p^{\mu-\lambda-1}I^{m,n+1}_{p_{i+1},q_{i+1},r}\left[\frac{1}{p}\left|\begin{matrix}\{(\mu-\lambda,1),(a_j,\alpha_j)_{1,n} & \ldots & (a_{ji},\alpha_{ji})_{n+1,p_i}\}\\ \{(b_j,\beta_j)_{1,m} & \ldots & (b_{ji},\beta_{ji})_{n+1,q_i}(-\lambda,1)\}\end{matrix}\right.\right] \qquad \ldots.4.2$$

Proof: We have [1, 1954; 5, 1960, 1997]

$$L[t^{\mu-1}(t+a)^{-\lambda}:p]=\Gamma(\mu)p^{(\lambda-\mu-1)/2}p^{(\lambda-\mu-1)/2}e^{ap/2}W_{k,m}(ap)$$

$$=\phi(ap) \qquad \ldots4.3$$

Where $k=\dfrac{(1-\mu-\lambda)}{2}, m=\dfrac{(\mu-\lambda)}{2}$

and $W_{k,m}$ is usual Whittaker function

$$L[(t-a)^{\mu-1}t^{-\lambda}H(t-a):p]=e^{-ap}\phi(ap) \qquad \ldots4.4$$

where H(t) is Heaviside's function

Now applying the operational pair (1.2) and (4.4) in the Parseval-Goldstein theorem, for Laplace transform and changing the order of integration we get

$$\int_a^{\infty}t^{-\lambda}(t-a)^{\mu-1}I(t)dt=\int_0^{\infty}t^{-\lambda}(t-a)^{\mu-1}g(t)H(t-a)dt$$

$$=\int_0^{\infty}e^{-at}\phi(at)I(t)dt \qquad \ldots4.5$$

Now substituting I (t) and $\phi(at)$ from (2.1) and (4.3) respectively in (4.5) and upon performing the indicated integration with the help of [1, 1954] we get the required results.

5. Special cases:

(i) By setting r =1 (2.1) reduces to

$$H_{p,q}^{m,n}[t] = \frac{1}{2\pi w}\int_L \left\{ \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + \beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j - \alpha_j s)} \right\} t^s ds \qquad \ldots 5.1$$

Equation (5.1) called Fox`s H- function [3, 1965] and in this case results reduces

$$W^{\mu}\{t^{-\lambda}H(t) : p\} = p^{\mu-\lambda-1}H_{p+1,q+1}^{m,n+1}\left[\frac{1}{p}\left| \begin{matrix} (\mu-\lambda,1),\ (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q},\quad (-\lambda,1) \end{matrix} \right.\right] \qquad \ldots 5.2$$

(ii) By setting r=1, $\alpha_j = 1, \beta_j = 1$ (2.1) reduces to

$$G_{p,q}^{m,n}[t] = \frac{1}{2\pi w}\int_L \left\{ \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} \right\} t^s ds \qquad \ldots 5.3$$

Equation (5.3) called Meijer`s G- function [4, 1946] and in this case results reduces

$$W^{\mu}\{t^{-\lambda}G(t) : p\} = p^{\mu-\lambda-1}G_{p+1,q+1}^{m,n+1}\left[\frac{1}{p}\left| \begin{matrix} \mu-\lambda, a_1 \ldots a_p \\ b_1 \ldots b_q, -\lambda \end{matrix} \right.\right] \qquad \ldots 5.4$$

Also considering particular values of parameters the results can be converted into Corresponding results in the form of series.

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